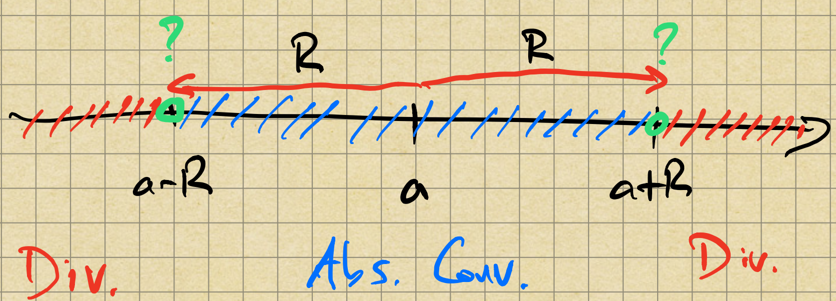
Power Series (Cont'd)

Recall: A <u>power series</u> is a function of the form:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

<u>**Theorem:**</u> There is an $0 \le R \le \infty$: radius of convergence



↳ <u>Interval of convergence</u>

(may/may not include endpoints)

<u>**Remarks:**</u> 1. We will usually find R using root or ratio test.

2. Endpoints must be considered separately, using <u>other</u> tests.

<u>Ex</u>: Find radius/interval of conv. of the following:

1. $$\sum_{n=1}^{\infty} \frac{(x-4)^n \cdot n}{3^n}$$

<u>Root Test</u>:

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-4)^n \cdot n}{3^n}\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{|x-4|^n \cdot n}{3^n}}$$

$$= \lim_{n\to\infty} \frac{|x-4| \cdot \sqrt[n]{n}}{3} = \frac{|x-4|}{3}$$

(Important to <u>know</u> $\lim_{n\to\infty} \sqrt[n]{n} = 1$)

<u>Conclusions</u>:

$|x-4| < 3$

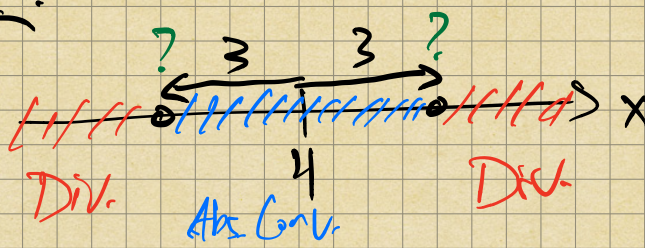
1. If $\frac{|x-4|}{3} < 1$, series conv. abs.

2. If $\frac{|x-4|}{3} > 1$, series div.

$$|x-4| > 3$$

$$\Rightarrow \boxed{R = 3}$$

Picture:



Endpoints:

x = 7: When we plug in we get

$$\sum_{n=1}^{\infty} \frac{(7-4)^n \cdot n}{3^n} = \sum_{n=1}^{\infty} n = \infty \quad \text{(Div.)}$$

by div. test.

↳ " $= -\frac{1}{12}$ "

We're @ the endpoints

~~Root/Ratio~~

~~Integral~~

x = 1: $\sum_{n=1}^{\infty} \frac{(1-4)^n \cdot n}{3^n} = \sum_{n=1}^{\infty} (-1)^n \cdot n$

~~Comparison~~ $\to$ there are neg. terms.

~~AST~~

$$(-3)^n = (-1\cdot 3)^n = (-1)^n \cdot 3^n$$

This diverges by div. test.

(AST cannot be used to prove divergence, only convergence [when it applies])

So interval of conv. is

$$\boxed{I = (1,7)}$$

2. $$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{(2(n+1))!} \Big/ \frac{(-1)^n x^{2n}}{(2n)!} \right|$$

$$= \lim_{n\to\infty} \frac{|x|^{2n+2}\,\cancel{(2n)!}}{|x|^{2n}\,(2n+2)!} = \lim_{n\to\infty} \frac{|x|^2}{(2n+2)(2n+1)}$$

$$\hookrightarrow (2n+2)(2n+1)\,\cancel{(2n)!}$$

$$= \frac{|x|^2}{\infty} = 0 < 1$$

$\Rightarrow$ Series conv. abs. for <u>all</u> x.

$$\Rightarrow \boxed{R = \infty, \quad I = (-\infty, \infty)}$$

<u>Visualizing P.S.</u>

$$\sum_{n=0}^{\infty} c_n (x-a)^n = \lim_{N\to\infty} \underbrace{\sum_{n=0}^{N} c_n (x-a)^n}$$

Can be graphed by a machine (it's just a poly.)

To visualize graph of $\sim$, we

plot the partial sums for increasing N and see what happens...

Properties of P.S.

Ex: 1. Recall that for $|x|<1$:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

This is diff. where series conv.!

2. Express $\frac{1}{3+x}$ as a P.S. centered @ $a=0$. What is radius?

$$\frac{1}{3+x} = \frac{1}{3}\cdot\frac{1}{\left(1+\frac{x}{3}\right)} = \frac{1}{3}\cdot\frac{1}{1-\left(-\frac{x}{3}\right)}$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-x}{3}\right)^n \quad \left|\frac{-x}{3}\right| < 1$$

Sum of geom. series

$$\Updownarrow$$

$$|x| < 3$$

$$\frac{1}{3+x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n+1}} \Rightarrow R = 3$$

Power Series (Cont.)

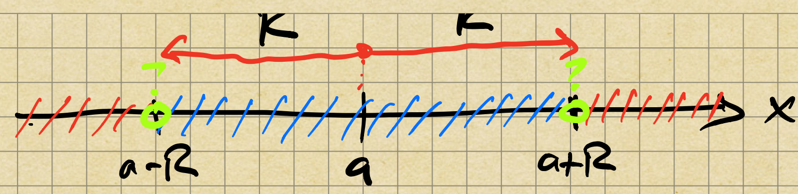
Recall: A P.S. is a function given by a series:

coeffs.

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

center

Theorem: Every P.S. has a radius of conv. $0 \leq R \leq \infty$ so that:



Div. $\underbrace{\text{Abs. Conv.}}$ Div. → the domain of P.S.

↳ Interval of conv.

Remarks:

1. To find R we typically use root/ratio test.

2. Endpoints must be handled by other tests.

Ex: Determine radius/interval of conv. for the following.

1. $$\sum_{n=1}^{\infty} \frac{(x-4)^n n}{3^n}$$

$\left[\text{P.S. w/ center } a = \underline{4} \text{ and coeffs. } c_n = \underline{n/3^n}\right]$

Root Test: $$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-4)^n n}{3^n}\right|}$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{|x-4|^n \cdot n}{3^n}} = \lim_{n\to\infty} \frac{|x-4|\cdot\sqrt[n]{n}}{3}$$

$$= \frac{|x-4|}{3}$$

(It's useful to <u>remember</u> that $\lim_{n\to\infty} \sqrt[n]{n} = 1$)

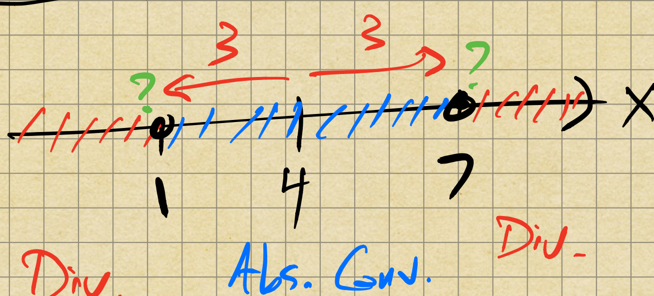
$|x-4|<3$

<u>Concl.</u> 1. $\frac{|x-4|}{3} < 1 \Rightarrow$ abs. conv.

2. $\frac{|x-4|}{3} > 1 \Rightarrow$ div.

$|x-4|>3$

<u>Picture</u>:



$\Rightarrow$ $R=3$

Endpoints:

$x=1$:

$$\sum_{n=1}^{\infty} \frac{(1-4)^n n}{3^n} = \sum_{n=1}^{\infty} \frac{(-3)^n n}{3^n}$$

$$= \sum_{n=1}^{\infty} (-1)^n \cdot n = -1+2-3+4-5+\cdots$$

Diverges

Possible Tests: ~~Root/Ratio~~

~~AST~~ (n does not dec. to 0)

~~Integral/Comparison~~ (series has neg. terms)

Start here. $\longrightarrow$ Div. Test: $(-1)^n \cdot n \not\to 0$

$x=7$:

$$\sum_{n=1}^{\infty} \frac{(7-4)^n n}{3^n} = \sum_{n=1}^{\infty} n \; \left(\text{"} = \frac{-1}{12} \text{"} \right)$$

Diverges by div. test $(u \not\to 0)$

$\Rightarrow$ Interval of conv. is

$$\boxed{I = (1, 7)}$$

2. $$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{(2(n+1))!} \Big/ \frac{(-1)^n x^{2n}}{(2n)!} \right|$$

$$= \lim_{n\to\infty} \frac{|x|^{2n+2}\,\cancel{(2n)!}}{|x|^{2n}(2n+2)!} \qquad (2n+2)\cdot(2n+1)\cancel{(2n)!}$$

$$= \lim_{n\to\infty} \frac{|x|^2}{(2n+2)(2n+1)} = \frac{|x|^2}{\infty} = 0 < 1$$

$\Rightarrow$ series conv. abs. for <u>all</u> x

$$\Rightarrow \boxed{R = \infty \ , \ I = (-\infty, \infty)}$$

<u>Visualizing P.S.</u>

$$\sum_{n=0}^{\infty} c_n (x-a)^n = \lim_{N \to \infty} \underbrace{\sum_{n=0}^{N} c_n (x-a)^n}$$

Cannot be "added" by a machine.

Partial sums are just polys., which <u>can</u> be computed mechanically.

<u>Moral</u>: By taking N large enough (and staying away from endpoints), the graph of partial sum will effectively be graph of series.

Properties of P.S.

1. Recall that for $|x|<1$:

$$\sum_{n=0}^{\infty} x^n = \boxed{\frac{1}{1-x}}$$

Know this!

So the P.S. is an inf. diff. function inside interval of convergence.

2. Notice that

Inf. diff. $\longrightarrow$ $\dfrac{1}{3+x} = \dfrac{1}{3}\cdot\dfrac{1}{1+x/3}$

$$= \frac{1}{3}\cdot\boxed{\frac{1}{1-(-x/3)}}$$

$\left|\frac{-x}{3}\right| < 1$

$\Updownarrow$

$|x| < 3$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-x}{3}\right)^n$$

$$\frac{1}{3+x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n+1}}$$ ← P.S. centred @ $a=0$ w/ radius $R=3$.

For $|x|<3$

<u>Def</u>: A function $f(x)$ is <u>analytic</u> at $a \in \mathbb{R}$ iff

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

w/ <u>positive</u> radius of conv.