Sequences

↳ (**Ordered**) **Infinite** List of Numbers:

$$\{a_1, a_2, a_3, \underset{\underset{\text{terms}}{\uparrow}}{a_4}, a_5, \ldots\} = \{a_n\}_{n=1}^{\infty}$$

with $a_n \in \mathbb{R}$.

Ex:

1. $$\{\overset{n=1}{1}, \overset{2}{1/2}, \overset{3}{1/3}, \overset{4}{1/4}, \overset{5}{1/5}, \overset{6}{1/6}, \overset{\cdots}{\ldots}\} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

$$a_n = n^{-1} = \frac{1}{n}$$

2. $$\{\overset{n=1}{1}, \xrightarrow[\times(-1)]{} \overset{2}{-1}, \xrightarrow[\times(-1)]{} \overset{3}{1}, \xrightarrow[\times(-1)]{} \overset{4}{-1}, \overset{5}{1}, \overset{6}{-1}, \overset{7}{1}, \overset{8}{-1}, \ldots\} = \{(-1)^{n+1}\}$$

$$a_n = (-1)^{\underline{\underline{n}}+1}$$

$$\Big\downarrow \times(-1)$$

$$\{-1, 1, -1, 1, -1, \ldots\} = \{(-1)^n\}$$

3. $$\{\overset{n=1}{1/2}, \overset{2}{2/3}, \overset{3}{3/4}, \overset{4}{4/5}, \overset{5}{5/6}, \ldots\}$$

$$= \left\{\frac{n}{n+1}\right\}$$

$$a_n = \frac{n}{n+1}$$

4. $\{1, \underline{1, 2}, \underline{3}, 5, 8, 13, 21, \ldots\}$

$$F_1 = F_2 = 1, \quad F_{n+1} = F_n + F_{n-1}$$

It turns out that

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

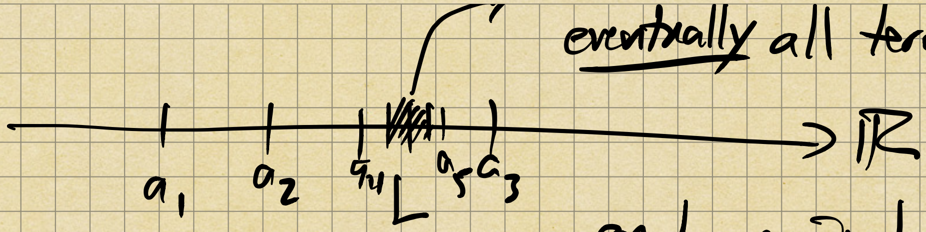
Convergence:

$\{a_n\}$ has limit L (converges to L) iff terms get and stay as close to L as we like.

Write: $\lim\limits_{n\to\infty} a_n = L$

Picture:

No matter how narrow,

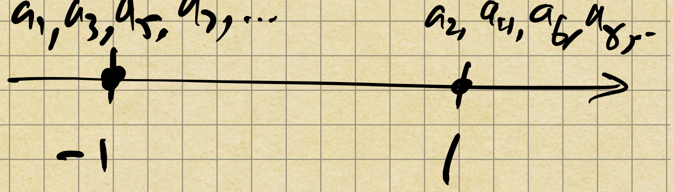


eventually all terms end up in here.

Ex:

1. $\lim_{n\to\infty} \frac{1}{n} = 0$

2. $\lim_{n\to\infty} (-1)^n$



DNE

3. $\lim_{n\to\infty} F_n \underset{\substack{\uparrow \\ \text{DNE}}}{=} \infty$

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\boxed{\frac{1+\sqrt{5}}{2}}\right)^n - \left(\underset{\substack{\downarrow \\ 0}}{\boxed{\frac{1-\sqrt{5}}{2}}}\right)^n\right)$$

Limit Laws:

$$\{a_n\}+\{b_n\}=\{a_n+b_n\}$$
$$c\{a_n\}=\{ca_n\}$$

If $\{a_n\}$, $\{b_n\}$ both converge and c is a constant:

1. $$\lim_{n\to\infty}\left(\underline{a_n}\pm\underline{b_n}\right)=\underline{\left(\lim_{n\to\infty}a_n\right)}\pm\underline{\left(\lim_{n\to\infty}b_n\right)}$$

2. $$\lim_{n\to\infty}(c\,a_n)=c\cdot\lim_{n\to\infty}a_n$$

3. $$\lim_{n\to\infty}(a_n b_n)=\left(\lim_{n\to\infty}a_n\right)\left(\lim_{n\to\infty}b_n\right)$$

4. $$\lim_{n\to\infty}\left(\boxed{a_n/b_n}\right)=\frac{\lim_{n\to\infty}a_n}{\boxed{\lim_{n\to\infty}b_n}}$$

provided $\lim_{n\to\infty}b_n\neq 0$.

5. If $f(x)$ is continuous at $x=L=\lim_{n\to\infty}a_n$, then

$$\lim_{n\to\infty}f(a_n)=f(L)=f\left(\lim_{n\to\infty}a_n\right)$$

Sequences

↳ (Ordered) infinite list of real #s:

$$\{\underbrace{a_1, a_2, a_3, a_4}_{\text{terms}}, \dots\} = \{a_{\underset{\text{index}}{n}}\}_{n=1}^{\infty}$$

Ex:

$$\begin{array}{lcccccccc} & n= & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1. & \{ & 1, & 1/2, & 1/3, & 1/4, & 1/5, & 1/6, & 1/7, \dots\} \end{array}$$

$$a_n = 1/n \qquad ↳ = \{1/n\}_{n=1}^{\infty}$$

$$2.\ \{1, \overset{\times(-1)}{\curvearrowright} -1, \overset{\times(-1)}{\curvearrowright} 1, \overset{\times(-1)}{\curvearrowright} -1, \overset{\times(-1)}{\curvearrowright} 1, \overset{\times(-1)}{\curvearrowright} -1, \dots\}$$

$$\begin{array}{ccccccc} n= & \overset{\uparrow}{1} & 2 & 3 & 4 & 5 & 6 \end{array} \qquad ↳ = \{(-1)^{n+1}\}_{n=1}^{\infty}$$

+2

$$a_n = \boxed{(-1)^{n-1}} = \underline{(-1)^{n+1}}$$

$(-1)^2 = 1$

$$\{-1, 1, -1, 1, -1, 1, \dots\} = \{(-1)^n\}_{n=1}^{\infty}$$

$$a_n = (-1)^n$$

3. $\{\underset{\downarrow\, n=1}{1}, 1/2, 1/4, 1/8, 1/16, 1/32, 1/64, \dots\}$

$$a_n = \frac{1}{2^{n-1}} \quad \hookrightarrow = \left\{\frac{1}{2^{\boxed{n-1}}}\right\}_{n=1}^{\infty}$$

$+1$ $\quad$ -1

$$= \left\{\frac{1}{2^{\boxed{n}}}\right\}_{n=0}^{\infty}$$

Shifting the index

4. Geometric Sequence:

$$\{1, r, r^2, r^3, r^4, r^5, \dots\}$$

$$= \{r^{n-1}\}_{n=1}^{\infty}$$

$$= \{r^n\}_{n=0}^{\infty}$$

#2, 3 are geom. w/ $r = -1, 1/2$, resp.

5. $$\begin{array}{cccccc} & \{1/2, & 2/3, & 3/4, & 4/5, & 5/6, \dots\} \\ n= & 1 & 2 & 3 & 4 & 5 \end{array}$$

$$a_n = \frac{n}{n+1} \quad \hookrightarrow = \left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty}$$

6. <u>Fibonacci Sequence</u>

$$\{1, 1, 2, 3, \underbrace{\boxed{5, 8}}_{+} \rightarrow 13, 21, 34, \dots\}$$

$$F_1 = F_2 = 1, \quad \underbrace{\boxed{F_{n+1} = F_n + F_{n-1}}}_{\text{recursion}}$$

<u>One can show</u>:

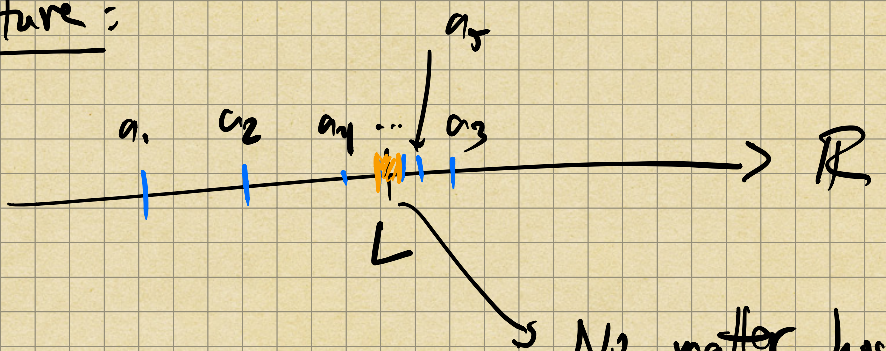
$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

Convergence

$\{a_n\}$ has limit L (converges to L) iff terms get and stay as close to L as we like.

Write: $\lim_{n\to\infty} a_n = L$ $\left(\lim \{a_n\} = L\right)$

Picture:

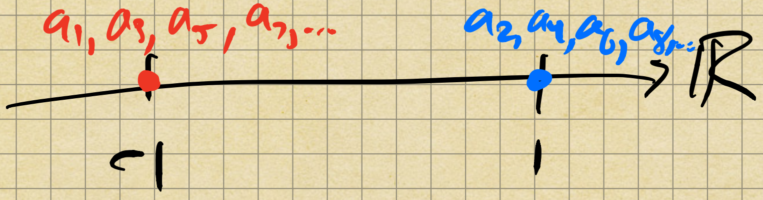


No matter how narrow, eventually all terms end up here.

Ex: 1. $\lim_{n\to\infty} \frac{1}{n} = 0$

↳ Archimedean Property of $\mathbb{R}$

2. $\lim_{n \to \infty} (-1)^n$ DNE



3. $\lim_{n \to \infty} F_n \underbrace{= \infty}_{\longrightarrow \text{DNE}}$

$\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}$

4. $r \in \mathbb{R}$,

$$\lim_{n \to \infty} r^n = \begin{cases} 1 & , r = 1 \\ \text{DNE} & , r = -1 \\ \text{DNE} & , |r| > 1 \\ & \hookrightarrow = \infty \text{ if } r > 1 \\ 0 & , |r| < 1 \end{cases}$$