Sequences (Cont.)

Ex: Use limit laws to evaluate.

1. $$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n}{n(1+1/n)}$$

$$= \lim_{n\to\infty} \frac{1}{1+1/n}$$

Limit Laws

$$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} 1/n}$$

$$= \frac{1}{1+0} = \boxed{1}$$

2. $$\lim_{n\to\infty} \frac{3n-5}{2n+1} = \lim_{n\to\infty} \frac{\cancel{n}(3-5/n)}{\cancel{n}(2+1/n)}$$

(Limit Laws) $$= \frac{3-0}{2+0} = \boxed{\frac{3}{2}}$$

3. $\lim_{n\to\infty} \frac{6n+5}{2n^3-8} = \lim_{n\to\infty} \frac{n(6+5/n)}{n^3(2-8/n^3)}$

$= \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{6+5/n}{2-8/n^3} = 0 \cdot \frac{6+0}{2-0}$

$= 0 \cdot 3$

$= \boxed{0}$

4. A <u>geometric</u> <u>sequence</u> has the form

$$\{1, r, r^2, r^3, r^4, r^5, \ldots\}$$

$$= \{r^n\}_{n=0}^{\infty}$$

$$\boxed{\lim_{n\to\infty} r^n = \begin{cases} 1, & r=1 \\ \text{DNE}, & r=-1 \\ 0, & |r|<1 \\ \text{DNE}, & |r|>1 \end{cases}}$$

5. $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = L$

$$\ln\left(\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n\right) = \ln L$$

$$\lim_{n\to\infty} \ln\left(\left(1+\frac{2}{n}\right)^n\right) = \ln L \quad \left(\text{Limit Laws}\right)$$

$$\lim_{n\to\infty} \underbrace{n}_{\infty}\cdot \underbrace{\ln\left(1+\frac{2}{n}\right)}_{0} = \ln L$$

$\infty \cdot 0$

$$\lim_{n\to\infty} \frac{\ln(1+2/n) \to 0}{1/n \to 0} = \ln L$$

L'Hopital

$$\lim_{n\to\infty} \frac{\left(\frac{1}{1+2/n}\right)\cdot\left(\frac{\cancel{-}2}{\cancel{n^2}}\right)}{\cancel{-}1/\cancel{n^2}} = \ln L$$

$$\lim_{n\to\infty} \frac{2}{1+2/n} = \ln L$$

Limit Laws

$$\frac{2}{1+0} = \ln L$$

$$2 = \ln L$$

$$\boxed{e^2 = L = \lim_{x\to\infty}\left(1+\frac{2}{n}\right)^n}$$

Consider:

$$143 = 1\cdot 10^2 + 4\cdot 10^1 + 3\cdot 10^0$$

$$\frac{1}{3} = 0.33333\ldots$$

$$= 3\cdot 10^{-1} + 3\cdot 10^{-2} + 3\cdot 10^{-3} + \cdots$$

Sequences (Cont.)

A **constant sequence** has the form

$$\{a, a, a, a, a, \ldots\}$$

Clearly

$$\lim_{n\to\infty} a = a.$$

Given sequences $\{a_n\}, \{b_n\}$ we set:

$$\{a_n\} \pm \{b_n\} = \{a_n \pm b_n\}$$

$$c\{a_n\} = \{ca_n\}$$

$$\{a_n\} \cdot \{b_n\} = \{a_n b_n\}$$

$$\{a_n\}/\{b_n\} = \{a_n/b_n\}$$

<u>Limit Laws</u>: Suppose $\{a_n\}, \{b_n\}$ are both convergent. Then:

1. $\lim_{n\to\infty}(a_n \pm b_n) = \left(\lim_{n\to\infty} a_n\right) \pm \left(\lim_{n\to\infty} b_n\right)$

2. $\lim_{n\to\infty}(ca_n) = c\cdot\left(\lim_{n\to\infty} a_n\right)$

3. $\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$

4. $\lim_{n\to\infty} \left(a_n / b_n\right) = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$

provided is not zero

5. If $f(x)$ is continuous at $x = L = \lim_{n\to\infty} a_n$, then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$$

Ex:

1. $$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n}{n(1+1/n)}$$

$$= \lim_{n\to\infty} \frac{1}{1+1/n}$$

Limit Laws

$$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}}$$

$$= \frac{1}{1+0} = 1$$

2. $$\lim_{n\to\infty} \frac{3n-5}{2n+1} = \lim_{n\to\infty} \frac{\cancel{n}(3-5/n)}{\cancel{n}(2+1/n)}$$

$$= \lim_{n\to\infty} \frac{3-5/n}{2+1/n}$$

Limit Laws

$$= \frac{3-0}{2+0}$$

$$= 3/2$$

3. $$\lim_{n\to\infty} \frac{6n+5}{2n^3-3n+1} = \lim_{n\to\infty} \frac{\cancel{n}(6+5/n)}{n^{\cancel{3}}(2-3/n^2+1/n^3)}$$

$$= \lim_{n\to\infty} \frac{1}{n^2}\cdot\frac{6+5/n}{2-3/n^2+1/n^3}$$

$$= 0\cdot\frac{6+0}{2-0+0} = 0\cdot 3 = \boxed{0}$$

4. $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = L$

$$\ln\left(\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n\right) = \ln L$$

Continuous!

Limit Law #5

$$\lim_{n\to\infty} \ln\left(1+\frac{2}{n}\right)^{n} = \ln L$$

$$\lim_{n\to\infty} n\cdot\ln\left(1+\frac{2}{n}\right) = \ln L$$

$\infty \cdot 0$

$$\lim_{n\to\infty} \frac{\ln(1+2/n)}{1/n} = \ln L$$

0

0

L'Hopital

$$\lim_{n\to\infty} \frac{\frac{1}{1+2/n}\cdot\left(\cancel{-}2/\cancel{n^2}\right)}{\cancel{-}1/\cancel{n^2}} = \ln L$$

$$\lim_{n\to\infty} \frac{2}{1+2/n} = \ln L$$

$$\frac{2}{1+0} = \ln L$$

$$2 = \ln L$$

$$\boxed{e^2 = L = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n}$$

<u>Remark</u>: Can use this technique to show:

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

Infinite Series:

Sequence: (Ordered) infinite list of #'s

Series: Sum of a sequence

$$\{a_1, a_2, a_3, a_4, \dots\} = \{a_n\}$$

$$a_1 + a_2 + a_3 + a_4 + \cdots = \sum_{n=1}^{\infty} a_n$$

Consider:

$$1397 = 1 \cdot 10^3 + 3 \cdot 10^2 + 9 \cdot 10^1 + 7 \cdot 10^0$$

$$\frac{1}{3} = 0.3333\ldots$$

$$= \underbrace{\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots}_{\text{Infinite Series!}}$$

$$1 = 0.9999\ldots$$

$$= \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{9}{10^n}$$