Exam 3
Q + A

13. $$\sum_{n=1}^{\infty} \frac{n!\,x^n}{7\cdot 15\cdot 23\cdot\cdots\cdot(8n-1)}$$

Ratio Test:

$$\lim_{n\to\infty}\left| \frac{(n+1)!\,x^{n+1}}{7\cdot 15\cdot 23\cdots(8(n+1)-1)} \Bigg/ \frac{n!\,x^n}{7\cdot 15\cdot 23\cdot\cdots\cdot(8n-1)}\right|$$

+8

$$=\lim_{n\to\infty} \frac{(n+1)!\,|x|^{\cancel{n+1}}\cdot \cancel{7}\cdot\cancel{15}\cdot\cancel{23}\cdot\cdots\cdot\cancel{(8n-1)}}{n!\,|x|^{\cancel{n}}\cdot\cancel{7}\cdot\cancel{15}\cdot\cancel{23}\cdot\cdots\cdot\cancel{(8n-1)}\cdot(8n+7)}$$

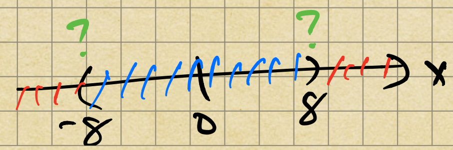
$$=\lim_{n\to\infty}\frac{|x|\cdot(n+1)}{8n+7} = \frac{|x|}{8}$$

Concl.:

Abs. if $\frac{|x|}{8} < 1 \iff |x| < 8$

Div. if $\frac{|x|}{8} > 1 \iff |x| > 8$

$\Rightarrow \boxed{R = 8}$



Interval of Conv.?

$\underline{x = \pm 8}$:

$$\sum_{n=1}^{\infty} \frac{n!\,(\pm 8)^n}{7 \cdot 15 \cdot 23 \cdot \cdots \cdot (8n-1)}$$

Series Diverges (Div. Test)

$$\boxed{\frac{n!\,8^n}{7 \cdot 15 \cdot 23 \cdots (8n-1)} \not\to 0}$$

Why?

$$\frac{n!\,8^n}{7 \cdot 15 \cdot 23 \cdots (8n-1)} = \frac{(1 \cdot 2 \cdot 3 \cdots n)(\overbrace{8 \cdot 8 \cdot 8 \cdots 8}^{n \text{ times}})}{7 \cdot 15 \cdot 23 \cdots (8n-1)}$$

$$= \left(\frac{8}{7}\right)\left(\frac{16}{15}\right)\left(\frac{24}{23}\right)\cdots\left(\frac{8n}{8n-1}\right) > 1$$

So interval is

$$\boxed{I = (-8, 8)}$$

14:

$$\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n} \Rightarrow \text{converges}$$

AST

$$f(n) = \frac{\ln n}{n}$$

$$f'(n) = \frac{n \cdot 1/n - \ln n \cdot 1}{n^2} = \frac{1 - \ln n}{n^2} < 0$$

for n large enough.

Terms decrease.

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

F16.1: Not for us today.

F13.11:

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$$

a)

$$S_n = \sum_{i=1}^{n} \left(\frac{1}{\sqrt{i}} - \frac{1}{\sqrt{i+1}} \right)$$

$$= \left(1 - \cancel{\frac{1}{\sqrt{2}}}\right) + \left(\cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{3}}}\right) + \left(\cancel{\frac{1}{\sqrt{3}}} - \cancel{\frac{1}{\sqrt{4}}}\right) +$$

$$\cdots + \left(\cancel{\frac{1}{\sqrt{n}}} - \frac{1}{\sqrt{n+1}} \right)$$

$$= \boxed{1 - \frac{1}{\sqrt{n+1}}}$$

b) $$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$$

$$= \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(1 - \frac{1}{\sqrt{n+1}} \right)$$

$$= \boxed{1}$$

Method of Undetermined Coefficients

Given

$$ay'' + by' + cy = e^{Ax} \left(p(x) \cos Bx + q(x) \sin Bx \right)$$

p, q polys. w/ deg. $\leq n$.

Given

$$y_p = e^{Ax}\left(P(x)\cos Bx + Q(x)\sin Bx\right)$$

Polys. w/ "undetermined" coeffs. of degree $\leq n$.

If any term in y_p solves

$$ay'' + by' + cy = 0,$$

replace y_p w/ $x y_p$. If this still has terms that solve homog. eqn., use $x^2 y_p$.

$r = 1, 2 \Rightarrow$ Char. poly. is

$$(r-1)(r-2) = r^2 - 3r + 2$$

$$y'' - 3y' + 2y = \boxed{e^{-x} \cdot x \cdot \sin 2x}$$

$y'' - 3y' + 2y = 0 \Rightarrow y = c_1 e^{x} + c_2 e^{2x}$

Guess:

$$y_p = e^{-x}\Big((Ax+B)\cos 2x + (Cx+D)\sin 2x \Big)$$

$$y_p'' - 3y_1' + 2y_p$$

$$\Big(\big((2x-5) + (-10x+4)C + B - 5D \big) \cos 2x$$

$$+ \Big(\big((10x-4)A + (2x-5)C + 10B + 2D \big) \cdot \sin 2x \Big) \Big)$$

$$= \quad x \cdot \sin 2x$$

$$(2x-5) + (-10x+4)C + B - 5D = 0$$

$$(10x-4)A + (2x-5)C + 10B + 2D = x$$

$$C_1 x + C_2$$

2. $\{b_n\}$

$$S_n = \sum_{j=1}^{n} b_j = 5 - \frac{2}{n}$$

a) $S_n - S_{n-1} = b_n \,, \quad n \geq 2$

$n=1$: $\boxed{b_1 = 5 - \frac{2}{1} = 3}$

b) $b_n = S_n - S_{n-1}$

$$= \left(\cancel{5} - \frac{2}{n}\right) - \left(\cancel{5} - \frac{2}{n-1}\right)$$

$$\boxed{b_n = \frac{2}{n-1} - \frac{2}{n} \qquad n \geq 2}$$

c) $\displaystyle\sum_{j=1}^{\infty} b_j = \lim_{n\to\infty} \sum_{j=1}^{n} b_j$

$$= \lim_{n\to\infty}\left(5 - \frac{2}{n}\right)$$

$$= \boxed{5}$$

<u>HW. 11</u>: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{b^n}(x-a)^n, \quad b > 0$

<u>Root Test</u>:

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n}{b^n}(x-a)^n\right|}$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{n}{b^n}|x-a|^n}$$

$$= \lim_{n\to\infty} \frac{\sqrt[n]{n}\,|x-a|}{b} = \frac{|x-a|}{b}$$

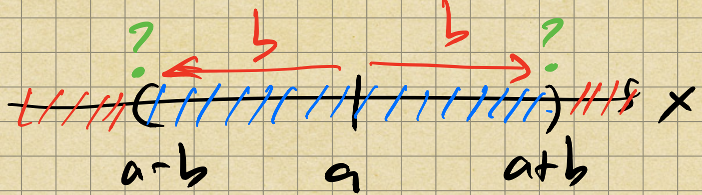
$$\boxed{\lim_{n\to\infty} \sqrt[n]{n} = 1}$$

Cond.:

Abs. conv. if $\frac{|x-a|}{b} < 1 \iff |x-a| < b$

Div. if $\frac{|x-a|}{b} > 1 \iff |x-a| > b$

$\Rightarrow \boxed{R = b}$



Interval of Conv.:

Endpoints:

$x = a \pm b$

$$\sum_{n=1}^{\infty} \frac{n}{\cancel{b^n}} (\cancel{a} \pm \cancel{b} - \cancel{a})^n = \sum_{n=1}^{\infty} (\pm 1)^n n$$

$$\frac{n(\pm b)^n}{b^n} = \frac{n(\pm 1 \cdot b)^n}{b^n}$$

$$= \frac{n(\pm 1)^n \cdot \cancel{b^n}}{\cancel{b^n}}$$

$(\pm 1)^n n \not\to 0 \Rightarrow$ series diverges by div. test.

Therefore

$$\boxed{I = (a-b, a+b)}$$

F11.3: $\displaystyle\sum_{n=2}^{\infty} e^{3-2n} = \sum_{n=2}^{\infty} e^3 e^{-2n}$

$$= e^3 \sum_{n=2}^{\infty} (e^{-2})^n = e^3 (e^{-2})^2 \sum_{n=0}^{\infty} (e^{-2})^n$$

$$(e^{-2})^2 + (e^{-2})^3 + (e^{-2})^4 + \cdots$$

"Standard" Geom. Series:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots$$

$$= \frac{1}{1-r} \quad (|r|<1)$$

$$\sum_{n=3}^{\infty} r^n = r^3 + r^4 + r^5 + r^6 + \cdots$$

$$= r^3(1 + r + r^2 + r^3 + \cdots)$$

$$= \frac{r^3}{1-r}$$

$$e^3(e^{-2})^2 \sum_{n=0}^{\infty} (e^{-2})^n = \boxed{e^{-1} \cdot \frac{1}{1-e^{-2}}}$$

mult. top/bottom

Provided $|e^{-2}| < 1$ ✓

$2y\,e$

$$= \boxed{\frac{e}{e^2-1}}$$

$$y'' + y' - 6y = e^{2x} \cdot 1$$

Solve comp. eqn. :

$$y'' + y' - 6y = 0$$

$$r^2 + r - 6 = 0$$

$$(r+3)(r-2) = 0$$

$$r = -3, 2$$

$\Rightarrow$ gen. sol'n is $y_h = c_1 e^{-3x} + c_2 e^{2x}$

Find particular sol'n :

Guess $y_p = Ae^{2x}$.

Modify: $y_p = Axe^{2x}$ ✓

$$y_p' = A(e^{2x} + 2xe^{2x})$$
$$= A(1+2x)e^{2x}$$

$$y_p'' = A(2 \quad + 2(1+2x) \quad)e^{2x}$$
$$= A(4+4x)e^{2x}$$

$$y_p'' + y_p' - 6y_p = \underline{A(4+4x)}e^{2x}$$
$$+ A(1+2x)e^{2x}$$
$$-6Axe^{2x} = e^{2x}$$

$$A\cancel{e^{2x}}(4 + \cancel{4x} + 1 + \cancel{2x} - \cancel{6x}) = \cancel{e^{2x}}$$

$$5A = 1 \Rightarrow A = 1/5$$

$$\Rightarrow \quad y_p = \frac{1}{5} x e^{2x}$$

Gen. Sol'n:

$$y = y_p + y_h$$

$$\boxed{y = \frac{1}{5} x e^{2x} + c_1 e^{-3x} + c_2 e^{2x}}$$

F13.7: $\sum_{n=1}^{\infty} a_n$ w/ partial sums

$$S_n = \frac{n}{n+1}.$$

Sum: $$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n$$

$$= \lim_{n \to \infty} \frac{n}{n+1} = \boxed{1}$$

Terms:

$$\boxed{a_1 = S_1 = \frac{1}{1+1} = \frac{1}{2}}$$

$$a_n = S_n - S_{n-1}$$

$$a_n + a_{n-1} + a_{n-2} + \cdots + a_1$$

$$-(a_{n-1} + a_{n-2} + \cdots + a_1)$$

$$= \frac{n}{n+1} - \frac{n-1}{(n-1)+1}$$

$$\boxed{a_n = \frac{n}{n+1} - \frac{n-1}{n} \quad (n \geq 2)}$$

$$= \frac{n^2 - (n-1)(n+1)}{(n+1)n}$$

$$= \frac{n^2 - (n^2-1)}{n(n+1)} = \frac{1}{n(n+1)}$$

F11.6: $\sum_{n=0}^{\infty} c_n (x-a)^n$

Given

$$\boxed{\lim_{n\to\infty} \sqrt[n]{|c_n|} = c \neq 0}$$

Root test:

$$\lim_{n\to\infty} \sqrt[n]{|c_n(x-a)^n|}$$

$$= \lim_{n\to\infty} \sqrt[n]{|c_n|} \cdot |x-a|$$

$$= c \cdot |x-a|$$

Concl.:

$|x-a| < \frac{1}{c} \iff c \cdot |x-a| < 1 \implies$ abs. conv.

$|x-a| > \frac{1}{c} \iff c \cdot |x-a| > 1 \implies$ div.

$$\implies R = 1/c \checkmark$$

F13,11: $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$

a) $S_n = \sum_{i=1}^{n} \left(\frac{1}{\sqrt{i}} - \frac{1}{\sqrt{i+1}} \right)$

$$= \left(\cancel{1} - \cancel{\frac{1}{\sqrt{2}}}\right) + \left(\cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{3}}}\right) + \left(\cancel{\frac{1}{\sqrt{3}}} - \cancel{\frac{1}{\sqrt{4}}}\right)$$

$$+ \cancel{\cdots} + \left(\cancel{\frac{1}{\sqrt{n}}} - \frac{1}{\sqrt{n+1}}\right)$$

$$\boxed{S_n = 1 - \frac{1}{\sqrt{n+1}}}$$

b) $$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{\sqrt{n+1}}\right) = \boxed{1}$$

2. $$\sum_{n=1}^{\infty} (-1)^{n+1} \boxed{\left(\sqrt{n^3+2} - \sqrt{n^3+1}\right)}$$

This is alternating.

decrease to 0? ✓

AST $\Rightarrow$ $\boxed{\text{converges}}$

$$\frac{\left(\sqrt{n^3+2}-\sqrt{n^3+1}\right)\left(\sqrt{n^3+2}+\sqrt{n^3+1}\right)}{\sqrt{n^3+2}+\sqrt{n^3+1}}$$

$$= \frac{(\cancel{n^3}+2)-(\cancel{n^3}+1)}{\sqrt{n^3+2}+\sqrt{n^3+1}}$$

$$= \frac{1}{\sqrt{n^3+2}+\sqrt{n^3+1}} \searrow 0. \checkmark$$

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \left(\quad \right) \right|$$

$$= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}+\sqrt{n^3+1}} \approx n^{3/2}$$

Do direct comparison w/ $\sum \frac{1}{n^{3/2}}$. → conv.

p-series w/ $p = 3/2$

$$\boxed{\frac{1}{\sqrt{n^3+2}+\sqrt{n^3+1}}} \le \frac{1}{\sqrt{n^3+2}}$$

$$\le \frac{1}{\sqrt{n^3}} = \boxed{\frac{1}{n^{3/2}}}$$

$\Rightarrow$ Abs. conv.

$\sum_{n=1}^{\infty} a_n$ conv. and $a_n > 0$.

$a_n \to 0^+$

Does $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converge?

Terms $\not\to 0$

$$\boxed{\frac{1}{a_n} \to \frac{1}{0^+} = \infty}$$

Diverges by div. test.