Derivatives + Integrals of Power Series

Theorem: Suppose $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of conv. $R > 0$. Then

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is continuous and differentiable on $(a-R, a+R)$ ($\Leftarrow$ interval of conv. w/o endpoints).

Furthermore:

$$1.\ f'(x) = \frac{d}{dx} \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$= \frac{d}{dx}\left(c_0 + c_1 (x-a)^1 + c_2 (x-a)^2 + \dots \right)$$

$$= \left(0 + c_1 + 2c_2 (x-a) + 3c_3 (x-a)^2 + \dots \right)$$

d/dx

We can diff. a conv. P.S. formally

$$= \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}$$

$$= \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

2. $$\int f(x)\,dx = \int \sum_{n=0}^{\infty} c_n (x-a)^n \,dx$$

$$= \sum_{n=0}^{\infty} \int c_n (x-a)^n \,dx$$

$$= \sum_{n=0}^{\infty} \frac{c_n (x-a)^{n+1}}{n+1} + C$$

Moreover, both $f'(x)$ and $\int f(x)\,dx$ have radius R. (Endpoint behavior may be different)

Ex: 1. $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x|<1$

$\downarrow d/dx$

$\dfrac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$ for $|x|<1$

For instance:

$$\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = 1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \cdots$$

$$= \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n-1}$$

$$= \frac{1}{(1-1/2)^2} = 4.$$

2. $\dfrac{1}{x} = \dfrac{1}{1+(x-1)} = \dfrac{1}{1-(1-x)}$

$$= \sum_{n=0}^{\infty} (1-x)^n, \quad |1-x| < 1$$

$$\Big\downarrow \int dx$$

$$\ln x = \sum_{n=0}^{\infty} \int (1-x)^n \, dx$$

$$= \sum_{n=0}^{\infty} \frac{-(1-x)^{n+1}}{n+1} + C$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (x-1)^{n+1}}{n+1} + C$$

$\Big\downarrow$ Replace n w/ $n-1$ (Reindexing)

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n} + C$$

To solve for C we plug in $x=1$ (center):

$$0 = \ln 1 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1-1)^n}{n} + C = 0+0+0+\cdots + C$$

$$= C$$

<u>So</u>:

$$\boxed{\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n} \quad \text{for } |x-1| < 1}$$

3. Recall that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

has $R = \infty$. So for any x:

$$f'(x) = \sum_{n=0}^{\infty} \frac{n x^{n-1}}{n!} \quad n! = n \cdot (n-1)!$$

$$= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

Replace n w/ n+1

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$$

i.e. $f(x)$ solves the ODE $y' = y$.

$$\Rightarrow f(x) = y = Ce^x$$

Plug in $x = 0$ (center) to find C:

$$\underline{\underline{1}} = \frac{0^0}{0!} = \sum_{n=0}^{\infty} \frac{0^n}{n!} = f(0) = Ce^0 = \underline{\underline{C}}$$

So:

$$\boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \quad \text{for } \underline{\text{all}} \ x}$$

In particular:

$$\boxed{e = e^1 = \sum_{n=0}^{\infty} \frac{1^n}{n!}} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$\approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{12!}$$

$$= 2.718281828\ldots$$

4. Find a P.S. representation for

$$\frac{1}{x^2 - 3x + 2}$$

centered @ $a = 0$.

Use the PFD:

$$\frac{1}{x^2-3x+2} = \frac{1}{(x-2)(x-1)} = \frac{1}{x-2} - \frac{1}{x-1}$$

$$= -\frac{1}{2}\cdot\frac{1}{1-(x/2)} + \frac{1}{1-x} \qquad \left|\frac{x}{2}\right| < 1 \text{ and } |x| < 1$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n + \sum_{n=0}^{\infty} x^n$$

$$= -\sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}} + \sum_{n=0}^{\infty} x^n$$

$$= \boxed{\sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+1}}\right)x^n = \frac{1}{x^2-3x+2} \quad \text{For } |x|<1}$$

5. Find a P.S. representatsen for

$\arctan x$ centered @ $a=0$.

$\Big\downarrow \, d/dx$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n, \quad |-x^2|<1$$

$$\Updownarrow$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad |x|<1$$

$\Big\downarrow \int dx$

$$\arctan x = \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + C$$

To find C we plug in $x=0$ (center):

$$\arctan 0 = \sum_{n=0}^{\infty} \frac{(-1)^n \, 0^{2n+1}}{2n+1} + C$$

$$0 = C$$

So we find:

$$\boxed{\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad \text{For } |x|<1}$$

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

In particular:

$$\lim_{x\to 1} \arctan x = \lim_{x \to 1} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

By Thm of Abel

$$\frac{\pi}{4} = \arctan 1 = \sum_{n=0}^{\infty} \lim_{x \to 1} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\boxed{\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}}$$

$$\left(\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots \right)$$

Differentiation + Integration of P.S.

Theorem: Suppose

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

has radius of convergence $R > 0$. Then $f(x)$ is diff. (hence continuous) on $(a-R, a+R)$ [$\leftarrow$ interval of conv.

w/o endpoints] and :

$$f'(x) = \frac{d}{dx} \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$= \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$\int f(x)\,dx = \int \sum_{n=0}^{\infty} c_n (x-a)^n \,dx$$

$$= \sum_{n=0}^{\infty} \frac{c_n (x-a)^{n+1}}{n+1} + C$$

and these have same radius R.

<u>Ex</u>: 1. $\displaystyle \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ <u>for $|x|<1$</u>

$\Big\downarrow \, d/dx$

$$\boxed{\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \text{ for } |x| < 1}$$

For instance, w/ $x = 1/2$:

$$\frac{1}{(1-1/2)^2} = \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^{n-1} = \sum_{n=1}^{\infty} \frac{n}{2^{n-1}}$$

$$\frac{1}{(1-1/2)^2} = 4$$

$$\Rightarrow \boxed{4 = \sum_{n=1}^{\infty} \frac{n}{2^{n-1}}}$$

$$= 1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \frac{5}{2^4} + \cdots$$

2. Find a P.S. representation for $\ln x$ centered @ $a = 1$.

$\Big|\ \frac{d}{d}$

$$\Big\downarrow \ d/dx$$

$$\frac{1}{x} = \frac{1}{1+(x-1)} = \frac{1}{1-(1-x)}$$

$|1-x| < 1$

$$= \sum_{n=0}^{\infty} (1-x)^n$$

$$\Big\downarrow \int dx$$

$$\ln x = \sum_{n=0}^{\infty} \frac{-(1-x)^{n+1}}{n+1} + C$$

for $|x-1|<1$

$$= \sum_{n=0}^{\infty} \frac{(-1)(-1)^{n+1}(x-1)^{n+1}}{n+1} + C$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1} + C$$

$$\big|$$

Replace n w/ $n-1$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n} + C$$

for $|x-1| < 1$

To find C we plug in $x = 1$ (center):

$$\ln 1 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\overset{0}{\cancel{(1-1)}}^n}{n} + C$$

$$0 = 0 + C = C$$

Therefore:

$$\boxed{\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n} \quad \text{for } |x-1| < 1}$$

$(0, 2)$

Thus:

$$\lim_{x\to 2^-} \ln x = \lim_{x\to 2^-} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n}$$

Thm. of Abel

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2-1)^n}{n}$$

$$\boxed{\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}}$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

3. Let $f(x) = \sum_{n=0}^{\infty} \boxed{\frac{x^n}{n!}} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

We know (via Ratio Test) this has $R = \infty$.

So for <u>any</u> x:

$\frac{d}{dx}$

$$f'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!}$$

$n! = n \cdot (n-1)!$

$$= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

Replace n w/ $n+1$

$$= \sum_{n=0}^{\infty} \frac{x^{n}}{n!} = f(x)$$

That is, $y = f(x)$ solves the ODE $y' = y$. So

$$f(x) = Ce^{x}$$

?

To find C, plug in $x = 0$:

$$\frac{\overset{1}{\cancel{0^0}}}{\underset{1}{\cancel{0!}}} = \sum_{n=0}^{\infty} \frac{0^{n}}{n!} = f(0) = Ce^{0} = C$$

$$\Rightarrow 1 = C$$

Thus: $$\boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \text{ for all } x.}$$

In particular, w/ $x=1$ we get:

$$e = e^1 = \sum_{n=0}^{\infty} \frac{1^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$\approx 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{12!}$$

$$\boxed{e = 2.718281828\ldots}$$

4. Find a P.S. for

$$\frac{1}{x^2 - 3x + 2}$$

centered @ $a = 0$.

Use the PFD:

$$\frac{1}{x^2-3x+2} = \frac{1}{(x-1)(x-2)} = \frac{1}{x-2} - \frac{1}{x-1}$$

$$= \boxed{\frac{-1}{2-x}} + \boxed{\frac{1}{1-x}}$$

$$= \frac{-1}{2(1-x/2)} + \frac{1}{1-x}$$

$|x| < 2$

$$= -\frac{1}{2} \cdot \frac{1}{1-x/2} + \frac{1}{1-x}$$

$|x/2| < 1$

and

$|x| < 1$

$$= -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n + \sum_{n=0}^{\infty} x^n$$

$$= -\sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}} + \sum_{n=0}^{\infty} x^n$$

$$= \boxed{\sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{n+1}}\right) x^n \text{ for } |x| < 1}$$

5. Find a P.S. representation for $\arctan x$ centered @ $a = 0$.

$$\arctan x \xrightarrow{d/dx} \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$\frac{1}{1-r}$$

$|x^2| < 1 \iff |x| < 1$

$$= \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\Big\downarrow \int dx$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + C$$

To solve for C we set $x = 0$ (center):

$$\arctan 0 = \sum_{n=0}^{\infty} \cancel{\frac{(-1)^n 0^{2n+1}}{2n+1}} + C$$

$\downarrow$

$$0 = 0 + C = C$$

Thus:

$$\boxed{\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}}$$

$|x|<1$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Notice:

$$\lim_{x\to 1^-} \arctan x = \lim_{x\to 1^-} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\frac{\pi}{4} = \arctan 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots}$$