Calc. II Final Exam

When: (5/12 (next Wed.))
Posted to Gradescope),
due 5/14 (Friday)

What: Cumulative

OH: Posted soon...

Old Exams: See TLEARN

Q: Find a P.S. for $f(x)$ (given)
centered @ $a=0$...

Idea: 1. Use algebra/calculus to relate
$f(x)$ to a function(s) resembling
the geom. series

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n, \quad |r|<1.$$

2. Use known series

#5: $f(x) = x^8 \arctan(x^3)$

#2 #1

We know:

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \le 1$$

So

$$\arctan(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{2n+1}, \quad |x^3| \le 1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{2n+1}$$

$$x^8 \cdot \arctan(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+11}}{2n+1}, \quad |x| \le 1$$

Taylor Series

Q: Given a "nice" $f(x)$, when can we write

$$\boxed{f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n}$$

for $|x-a| < R$.

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

$$f''(x) = 2c_2 + 3!c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \cdots$$

$$f'''(x) = 3!c_3 + 4!c_4(x-a) + \cdots$$

$$\vdots$$

So:

$$f(a) = c_0 \quad \Rightarrow \quad c_0 = f(a)/0!$$

$$f'(a) = c_1 \quad \Rightarrow \quad c_1 = f'(a)/1!$$

$$f''(a) = 2c_2 \quad \Rightarrow \quad c_2 = f''(a)/2!$$

$$f'''(a) = 3!\,c_3 \implies c_3 = \frac{f'''(a)}{3!}$$

$$\vdots$$

$$f^{(n)}(a) = n!\,c_n \implies c_n = \frac{f^{(n)}(a)}{n!}$$

$$\vdots$$

So <u>if</u> $f(x)$ is given by a P.S. centered @ a, then it <u>must</u> be

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

↳ <u>Taylor series</u> for $f(x)$ @ $x=a$.

<u>Q'</u>: How can we tell if $f(x)$ is actually equal to its T.S.?

<u>A</u>: Suppose $f \in C^\infty([a,b])$.

Start w/ FTOC: for any $x \in [a, b]$:

$$\int_a^x f'(t)\,dt = f(x) - f(a)$$

$$f(x) = f(a) + \int_a^x f'(t)\,dt$$

Now integrate by parts:

$$\begin{array}{ccc} f'(t) & \ominus & dt \\ \hline f''(t) & \ominus & x-t \\ f'''(t) & \ominus & (x-t)^2/2 \\ \vdots & \vdots & \vdots \\ \int f^{(n)}(t) & \oplus & (x-t)^{n-1}/(n-1)! \end{array}$$

$$\int dt = t - x$$

$$f(x) = f(a) - f''(t)\frac{(x-t)^2}{2}\Big|_a^{x=t} - \frac{f'''(t)(x-t)^3}{3!}\Big|_a^{x=t}$$

$$- f'(t)(x-t)\Big|^x$$

$$- \cdots \quad + \int_a^x f^{(n)}(t)\,\frac{(x-t)^{n-1}}{(n-1)!}\,dt$$

$$f(x) = \underbrace{f(a) + \overset{f'(a)(x-a)}{\vee} + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}}_{\text{Partial sum of T.S.}} \; \underbrace{+ \int_a^x f^{(n)}(t)\,\frac{(x-t)^{n-1}}{(n-1)!}\,dt}_{\text{Remainder}}$$

Partial sum of T.S.

Remainder

Taylor's Theorem w/ Remainder → See below for a more careful statement.

<u>Moral</u>: If rem. $\to 0$ as $n \to \infty$,

then $f(x) = $ T. S.

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#6:

$$f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

Recall:

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}, \quad |x| < 1$$

$$\ln(1-x) = \ln(1+(-x))$$

$|-x| < 1$

$|x| < 1$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-x)^n}{n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot (-1)^n x^n}{n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{2n+1} x^n}{n}$$

odd #

$=$

$$\ln(1+x) - \ln(1-x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} - \left(-\sum_{n=1}^{\infty} \frac{x^n}{n}\right)$$

$$= \sum_{n=1}^{\infty} \boxed{\frac{(-1)^{n+1} x^n}{n}} + \sum_{n=1}^{\infty} \boxed{\frac{x^n}{n}}$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{\boxed{\left((-1)^{n+1}+1\right)} x^n}{n} = \sum_{m=0}^{\infty} \frac{2 \cdot x^{2m+1}}{2m+1}}$$

$\begin{cases} 2 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

For $|x|<1$.

Taylor Series

Q: Given $f(x)$, when can we write

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \text{ ?}$$

for $|x-a|<R$

A: At least we must have $f(x)$ inf. diff. (since every P.S. is).

Suppose :

a

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

$$f''(x) = 2c_2 + 3!\,c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \cdots$$

$$f'''(x) = 3!\,c_3 + 4!\,c_4(x-a) + 5\cdot 4\cdot 3c_5(x-a)^2 + \cdots$$

$$\vdots$$

Plug in $x = a$:

$$\left.\begin{aligned} f(a) &= c_0 \\ f'(a) &= c_1 \\ f''(a) &= 2c_2 \\ f'''(a) &= 3!\,c_3 \\ &\vdots \\ f^{(n)}(a) &= n!\,c_n \end{aligned}\right\} \Rightarrow \begin{aligned} c_0 &= f(a) \\ c_1 &= f'(a) \\ c_2 &= \frac{f''(a)}{2} \\ c_3 &= \frac{f'''(a)}{3!} \\ &\vdots \\ c_n &= \frac{f^{(n)}(a)}{\;} \end{aligned}$$

$n!$

Theorem: If $f(x)$ equals a P.S. centered @ $x=a$ (w/ positive radius), then

$$\boxed{f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n}$$

$\rightarrow$ Taylor series for $f(x)$ @ $x=a$

Moral: If $f(x)$ has a P.S. expansion, it is unique.

Q': When is $f(x)$ equal to its T.S.?

inf. fns. on $[a,b]$

Suppose $f(x)\in C^{\infty}([a,b])$. Then

FTOC $\Rightarrow \displaystyle\int_a^x f'(t)\,dt = f(t)\Big|_{t=a}^{t=x}$

$$= f(x) - f(a)$$

$$\Rightarrow \quad f(x) = f(a) + \underline{\int_a^x f'(t)\,dt}$$

Now integrate by parts:

$$\begin{array}{ccc} f'(t) & - & 1\cdot dt \\ \hline f''(t) & - & x-t \\ f'''(t) & - & (x-t)^2/2 \\ f^{(4)}(t) & - & (x-t)^3/3! \\ \vdots & & \vdots \\ \underline{f^{(n)}(t)} & \xrightarrow[\int dt]{} & (x-t)^{(n-1)}/(n-1)! \end{array}$$

$$\int dt = t + C$$

$-x$

So we have

$$f(x) = f(a) + f'(t)(x-t)\Big|_{t=x}^{t=a}$$

$$+ \frac{f''(t)(x-t)^2}{2}\Big|_{t=x}^{t=a} + \frac{f'''(t)(x-t)^3}{3!}\Big|_{t=x}^{t=a}$$

$$+ \cdots + \frac{f^{(n-1)}(t)(x-t)^{n-1}}{(n-1)!}\Big|_{t=x}^{t=a}$$

$$+ \int_a^x f^{(n)}(t)\,\frac{(x-t)^{n-1}}{(n-1)!}\,dt$$

$$f(x) = \boxed{f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!}}$$

$$\boxed{+ \int_a^x f^{(n)}(t)\,\frac{(x-t)^{n-1}}{(n-1)!}\,dt}$$

Partial sum of T.S.!

$\hookrightarrow$ Remainder.

Taylor's Theorem w/ Remainder

Moral: To show $f(x) = T.S.$, must show remainder $\to 0$ as $n \to \infty$.

Stated more carefully, Taylor's Theorem is:

Theorem: Let I be an open interval w/ $a \in I$. If $f(x) \in C^{\infty}(I)$, then for any $x \in I$ and any $n \in \mathbb{N}_0$ one has:

$$f(x) = \boxed{\sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!}(x-a)^i} + \boxed{\frac{1}{(n-1)!}\int_a^x f^{(n)}(t)(x-t)^{n-1}\,dt}$$

$\hookrightarrow (n-1)^{st}$ partial sum of T.S.

$\hookrightarrow$ The <u>remainder</u> $R_n(x)$.

<u>Corollary</u>: Under the hypotheses above:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \iff \lim_{n\to\infty} R_n(x) = 0.$$

<u>Remark</u>: This condition is not as unreasonable as it might look. Recall that

$$R_n(x) = \frac{1}{(n-1)!}\int_a^x f^{(n)}(t)\,(x-t)^{n-1}\,dt$$

Unless the derivatives of $f(x)$ grow <u>very</u> rapidly, the $\frac{1}{(n-1)!}$ will usually "kill" the remainder (as $n \to \infty$).