Improper Integrals of Type II

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Calculus II

Recall

To evaluate an integral of the form

$$\int_a^\infty f(x)\,dx$$

we replaced the unbounded interval $[a,\infty)$ with the closed interval $[a,t]$, then let $t \to \infty$.

We can perform an analogous procedure to evaluate the definite integral of a function $f(x)$ with a discontinuity.

That is, we replace the interval of integration with one that "avoids" the discontinuity, then take a limit.

Motivating Example

Suppose we are asked to find the area between the curve $y = 1/x^2$ and the x-axis, for $-1 \leq x \leq 1$.

That is, we want to compute $\displaystyle\int_{-1}^{1} \frac{dx}{x^2}$.

Appealing to FTOC we have

$$\int_{-1}^{1} \frac{dx}{x^2} = \frac{-1}{x}\bigg|_{-1}^{1} = -1 - 1 = -2.$$

Clearly something has gone wrong, since $y = 1/x^2$ is always *above* the x-axis.

Improper Integrals of Type II

Problem. $f(x) = \frac{1}{x^2}$ is *not* continuous on $[-1, 1]$ (why?), so FTOC *does not apply*!

As with an integral over an unbounded domain, we can deal with a discontinuous integrand by taking an appropriate limit.

Definition (Improper Integral of Type II)

Suppose $f(x)$ is continuous on $[a, b)$. We define

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx,$$

provided the limit exists. In this case we say the improper integral *converges*. Otherwise we say it *diverges*.

Remarks

If $f(x)$ is instead continuous on $(a, b]$, we set

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx.$$

An improper integral of Type I can be thought of as an improper integral of Type II with $b = \infty$ or $a = -\infty$.

If $f(x)$ is actually continuous throughout $[a, b]$, then $|f(x)|$ has a maximum value M there. So as $t \to b^-$ we have

$$\left|\int_t^b f(x)\,dx\right| \le \int_t^b |f(x)|\,dx \le M(b-t) \to 0.$$

Thus

$$\int_a^t f(x)\,dx = \int_a^b f(x)\,dx - \int_t^b f(x)\,dx \to \int_a^b f(x)\,dx.$$

So our definition of an improper integral of Type II agrees with the "proper" integral when the integrand is continuous throughout $[a, b]$.

Example 1

Evaluate $\displaystyle\int_4^5 \frac{dx}{\sqrt{x-4}}$ or show that it diverges.

Solution. The integrand is continuous on $(4, 5]$ so we have

$$\int_4^5 \frac{dx}{\sqrt{x-4}} = \lim_{t\to 4^+} \int_t^5 \frac{dx}{\sqrt{x-5}} = \lim_{t\to 4^+} 2\sqrt{x-4}\Big|_t^5$$

$$= \lim_{t\to 4^+} 2 - 2\sqrt{t-4} = \boxed{2}.$$

Example 2

Evaluate $\int_0^3 \frac{dx}{\sqrt[3]{3-x}}$ or show that it diverges.

Solution. The integrand is continuous on $[0,3)$, so we have

$$\begin{aligned}\int_0^3 \frac{dx}{\sqrt[3]{3-x}} &= \lim_{t\to 3-} \int_0^t \frac{dx}{\sqrt[3]{3-x}} \\ &= \lim_{t\to 3-} \left.\frac{-(3-x)^{2/3}}{2/3}\right|_0^t \\ &= \lim_{t\to 3^-} \frac{3}{2}\left(3^{2/3}-(3-t)^{2/3}\right) = \boxed{\frac{3^{5/3}}{2}}.\end{aligned}$$

□

Interior Discontinuities

If an integrand is discontinuous at an *interior* point, we must split the integral at that point and compute two improper integrals.

Example 3

Evaluate $\int_{-1}^{1} \frac{dx}{x^2}$ or show that it diverges.

Solution. The integrand is discontinuous at $0 \in (-1, 1)$, so we must set

$$\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2}$$

and compute the two improper integrals separately.

Since

$$\int_0^1 \frac{dx}{x^2} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x^2} = \lim_{t\to 0^+} \frac{-1}{x}\bigg|_t^1 = \lim_{t\to 0^+} \frac{1}{t} - 1 = \infty,$$

we conclude that the overall integral diverges. □

We can treat "doubly improper" integrals the same way.

Example 4

Evaluate $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$ or show that it diverges.

Solution. Because the integral is improper at both ends, we set

$$\int_0^\infty \frac{dx}{\sqrt{x}(x+1)} = \int_0^1 \frac{dx}{\sqrt{x}(x+1)} + \int_1^\infty \frac{dx}{\sqrt{x}(x+1)}$$

and evaluate each of these as a limit.

To antidifferentiate we substitute $x = u^2$, $dx = 2u\,du$:

$$\int \frac{dx}{\sqrt{x}(x+1)} = \int \frac{2u\,du}{u(u^2+1)} = 2\arctan u + C = 2\arctan\sqrt{x} + C.$$

Thus

$$\begin{aligned}
\int_0^\infty \frac{dx}{\sqrt{x}(x+1)} &= \lim_{t\to 0^+}\int_t^1 \frac{dx}{\sqrt{x}(x+1)} + \lim_{s\to\infty}\int_1^s \frac{dx}{\sqrt{x}(x+1)} \\
&= \lim_{t\to 0^+} 2\arctan\sqrt{x}\Big|_t^1 + \lim_{s\to\infty} 2\arctan\sqrt{x}\Big|_1^s \\
&= \lim_{t\to 0^+}\left(2\arctan(1) - 2\arctan(\sqrt{t})\right) \\
&\quad + \lim_{s\to\infty}\left(2\arctan(\sqrt{s}) - 2\arctan(1)\right) \\
&= 2\arctan(1) + \pi - 2\arctan(1) = \boxed{\pi}.
\end{aligned}$$

Example 5

Evaluate $\displaystyle\int_{-2}^{2} \frac{x}{x^2 - 3x - 4}\, dx$ or show that it diverges.

Solution. Since

$$x^2 - 3x - 4 = (x-4)(x+1),$$

the integrand is discontinuous at $-1 \in (-2, 2)$. So we set

$$\int_{-2}^{2} \frac{x}{x^2-3x-4}\, dx = \int_{-2}^{-1} \frac{x}{x^2-3x-4}\, dx + \int_{-1}^{2} \frac{x}{x^2-3x-4}\, dx$$

and take limits.

The PFD of the integrand is

$$\frac{x}{x^2-3x-4} = \frac{4/5}{x-4} + \frac{1/5}{x+1},$$

so that

$$\begin{aligned}
\int_{-2}^{-1} \frac{x}{x^2-3x-4}\,dx &= \lim_{t\to -1^-} \int_{-2}^{t} \frac{4/5}{x-4} + \frac{1/5}{x+1}\,dx \\
&= \lim_{t\to -1^-} \left(\frac{4}{5}\ln|x-4| + \frac{1}{5}\ln|x+1| \Big|_{-2}^{t} \right) \\
&= \lim_{t\to -1^-} \frac{4}{5}\ln|t-4| + \frac{1}{5}\ln|t+1| - \frac{4}{5}\ln 6 \\
&= -\infty
\end{aligned}$$

which means the original integral diverges. □

Example 6

Evaluate $\int_0^1 x \ln x \, dx$ or show that it diverges.

Solution. The integrand is undefined at $x = 0$ so we set

$$\int_0^1 x \ln x \, dx = \lim_{t \to 0^+} \int_t^1 x \ln x \, dx.$$

If we integrate by parts with $u = \ln x$ and $dv = x \, dx$ we find

$$\lim_{t \to 0^+} \int_t^1 x \ln x \, dx = \lim_{t \to 0^+} \left(\left. \frac{x^2 \ln x}{2} \right|_t^1 - \int_t^1 \frac{x}{2} \, dx \right)$$

$$= \lim_{t\to 0^+} \left(-\frac{x^2 \ln x}{2} - \frac{x^2}{4}\Big|_t^1 \right)$$

$$= \lim_{t\to 0^+} \left(-\frac{1}{4} + \frac{t^2 \ln t}{2} + \frac{t^2}{4} \right)$$

$$= -\frac{1}{4} + \lim_{t\to 0^+} \left(\frac{t^2 \ln t}{2} \right) = -\frac{1}{4} + \lim_{t\to 0^+} \frac{\ln t}{2t^{-2}}$$

$$= -\frac{1}{4} + \lim_{t\to 0^+} \frac{1/t}{-4t^{-3}} = -\frac{1}{4} + \lim_{t\to 0^+} \frac{t^2}{-4}$$

$$= \boxed{-\frac{1}{4}}.$$

$\square$

Example 7

For what values of $q > 0$ does $\int_0^1 \frac{dx}{x^q}$ converge?

Solution. If we substitute $u = 1/x$, $du = -dx/x^2$, then as $x \to 0^+$ we have $u \to \infty$.

Thus

$$\int_0^1 \frac{dx}{x^q} = \int_\infty^1 \frac{-du/u^2}{u^{-q}} = \int_1^\infty \frac{du}{u^{2-q}}.$$

We know that this integral converges if and only if $p = 2 - q > 1$ or

$$\boxed{q < 1}.$$

□