Introduction to Ordinary Differential Equations

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Calculus II

Introduction

In numerous applications, a given quantity is most naturally described through a relationship (equation) involving its derivative(s).

For instance, Newton's second law ($F = ma$) gives the relationship between the forces acting on an object, its mass, and its acceleration, which is the second derivative of its position.

Equations that relate a function to its derivative(s) are called *differential equations*, and learning how to solve certain types of differential equations will be our goal for the next few weeks.

Ordinary Differential Equations

An *ordinary differential equation* (ODE) is an equation of the form

$$F(x, y, y', y'', \dots, y^{(n)}) = 0,$$

where $y = y(x)$ is an unknown function of the (independent) variable x.

Remark. It is common (and sometimes useful) to write some ODEs with nonzero terms on *both* sides of the equality.

For instance, the ODE $y^2 + y' = 0$ is equivalent to $y' = -y^2$.

Examples of ODEs

1. $y' = -y^2$
2. $(\sin x)\dfrac{dy}{dx} + (\cos x)y = x$
3. $y' = 3x^2 + 5x$
4. $\dfrac{dP}{dt} = kP$, where k a constant (the *natural growth equation*)
5. $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{M}\right)$, where k, M are constants (the *logistic equation*)
6. $\dfrac{d^2x}{dt^2} = -\dfrac{k}{m}x$ where k, m are positive constants (the *spring motion equation*)
7. $y'' - 5y' + 6y = 6x + 1$

Terminology

The *order* of an ODE is the highest derivative occurring in the equation.

Examples 1-5 above all have order 1 (or are *first order*). Examples 6 and 7 have order 2 (are *second order*).

A *solution* to an ODE (with independent variable x) is a function $f(x)$ so that the ODE is true when we set $y = f(x)$.

In principle, one can always check if a given function is a solution to an ODE: simply plug it in and see if the equation is valid!

Examples

Example 1

Show that for any constant C the function

$$y = \frac{1}{x + C}$$

is a solution to $y' = -y^2$.

Solution. We have

$$y' = \frac{-1}{(x + C)^2} = -\left(\frac{1}{x + C}\right)^2 = -y^2.$$

□

Remark. We will see that, together with $y = 0$, this gives *every* possible solution to $y' = -y^2$.

Example 2

Show that for any constant C the function

$$y = \frac{x^2 + C}{2 \sin x}$$

is a solution to $(\sin x)\dfrac{dy}{dx} + (\cos x)y = x$.

Solution. The quotient rule yields

$$\frac{dy}{dx} = \frac{4x \sin x - 2(x^2 + C) \cos x}{4 \sin^2 x},$$

so that

$$\begin{aligned}(\sin x)\frac{dy}{dx} + (\cos x)y &= \frac{4x \sin x - 2(x^2 + C) \cos x}{4 \sin x} + \frac{(\cos x)(x^2 + C)}{2 \sin x} \\ &= \frac{4x \sin x}{4 \sin x} = x.\end{aligned}$$

□

Example 3

Find every solution to the ODE $y' = 3x^2 + 5x$.

Solution. Integrating both sides we have

$$y = \int y'\,dx = \int 3x^2 + 5x\,dx = \boxed{x^3 + \frac{5}{2}x + C}.$$

□

Remark. In general, the solutions to $y' = f(x)$ are given by

$$y = \int f(x)\,dx.$$

So finding antiderivatives amounts to solving (very simple) ODEs!

Example 4

Show that for any constant P_0 the function

$$P = P_0 e^{kt}$$

solves the natural growth equation $\dfrac{dP}{dt} = kP$.

Solution. If $P = P_0 e^{kt}$, then the chain rule gives

$$\frac{dP}{dt} = P_0 k e^{kt} = kP.$$

□

Remark. We will see that this gives *every* solution to the natural growth equation.

Example 5

Show that for any constant C the function

$$P = \frac{M}{1 + Ce^{-kt}}$$

solves the logistic equation $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{M}\right)$.

Solution. Writing $P = M(1 + Ce^{-kt})^{-1}$, the chain rule yields

$$\begin{aligned}
\frac{dP}{dt} &= -M(1 + Ce^{-kt})^{-2}(-Cke^{-kt}) = \frac{MCke^{-kt}}{(1 + Ce^{-kt})^2} \\
&= k \cdot \frac{M}{1 + Ce^{-kt}} \cdot \frac{Ce^{-kt}}{1 + Ce^{-kt}} = kP \cdot \frac{Ce^{-kt} + 1 - 1}{1 + Ce^{-kt}} \\
&= kP\left(1 - \frac{1}{1 + Ce^{-kt}}\right) = kP\left(1 - \frac{P}{M}\right).
\end{aligned}$$

□

Example 6

Show that for any constants C_1 and C_2 the function

$$x(t) = C_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$$

is a solution of the spring motion equation $\dfrac{d^2x}{dt^2} = -\dfrac{k}{m}x$.

Solution. By the chain rule we have

$$\frac{dx}{dt} = -C_1\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}t\right) + C_2\sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}t\right)$$

so that

$$\frac{d^2x}{dt^2} = -C_1\frac{k}{m}\cos\left(\sqrt{\frac{k}{m}}t\right) - C_2\frac{k}{m}\sin\left(\sqrt{\frac{k}{m}}t\right) = -\frac{k}{m}x.$$

□

Example 7

Show that for any constants C_1 and C_2 the function

$$y = C_1 e^{2x} + C_2 e^{3x} + x + 1$$

is a solution to $y'' - 5y' + 6y = 6x + 1$.

Solution. If $y = C_1 e^{2x} + C_2 e^{3x} + x + 1$, then

$$y' = 2C_1 e^{2x} + 3C_2 e^{3x} + 1 \;\Rightarrow\; y'' = 4C_1 e^{2x} + 9C_2 e^{3x}.$$

Thus

$$\begin{aligned} y'' - 5y' + 6y = (4C_1 e^{2x} + 9C_2 e^{3x}) - 5(2C_1 e^{2x} + 3C_2 e^{3x} + 1) \\ + 6(C_1 e^{2x} + C_2 e^{3x} + x + 1) = 6x + 1. \end{aligned}$$

□

Although it is (usually) easy to check that a given function solves a certain ODE, this will rarely be our concern.

Instead, we will typically be given an ODE and asked to *find* its solutions.

Solving an ODE means finding *every* solution. Typically there are infinitely many solutions, but frequently they can be described in terms of one or more parameters (constants).

Solving ODEs in general is quite difficult. However, if we restrict the type of ODE being considered (e.g. first order linear ODEs), we can sometimes give explicit solution procedures.

Initial Value Problems

An *initial value problem* (IVP) is an ODE of order n together with *initial conditions*

$$y(x_0) = c_0, \;\; y'(x_0) = c_1, \;\; y''(x_0) = c_2, \ldots, y^{(n-1)}(x_0) = c_{n-1}.$$

A *solution* to an IVP is a function that solves the ODE *and* satisfies the initial conditions.

While ODEs have many solutions, the solutions to IVPs are typically unique.

To solve an IVP one usually solves the ODE first, then determines the values of the parameters in the solution so that the initial conditions are satisfied.

Example 8

Solve the IVP

$$y' = -y^2, \quad y(1) = 5.$$

Solution. According to the remark following example 1, the ODE $y' = -y^2$ has the *general solution*

$$y = \frac{1}{x + C}.$$

We simply need to choose C so that $y(1) = 5$:

$$5 = y(1) = \frac{1}{1 + C} \;\Rightarrow\; 1 + C = \frac{1}{5} \;\Rightarrow\; C = -\frac{4}{5}.$$

Therefore the *particular solution* is

$$\boxed{y = \frac{1}{x - 4/5}}.$$

□

Example 9

Solve the IVP

$$y'' - 5y' + 6y = 6x + 1, \quad y(0) = -1, \quad y'(0) = 1.$$

Solution. The general solution was given in Example 7:

$$y = C_1 e^{2x} + C_2 e^{3x} + x + 1 \;\Rightarrow\; y' = 2C_1 e^{2x} + 3C_2 e^{3x} + 1.$$

The initial conditions require that

$$-1 = y(0) = C_1 + C_2 + 1 \quad \text{and} \quad 1 = y'(0) = 2C_1 + 3C_2 + 1.$$

Variable elimination yields $C_1 = -6$ and $C_2 = 4$. Thus

$$\boxed{y = -6e^{2x} + 4e^{3x} + x + 1}.$$

□

Slope Fields

A first order ODE of the form

$$\frac{dy}{dx} = F(x, y) \tag{1}$$

can be though of as assigning a slope to each point in the *xy*-plane.

This can be visualized by drawing a small line segment of slope $F(x, y)$ at each point (x, y).

The resulting diagram is called a *slope field* or *direction field*.

The solutions to (1) are the functions whose graphs are tangent to the slope field at each point.

Slope fields can provide useful qualitative information about the solutions to (1) without the need to actually solve it first.

Remarks.

- Only first order ODEs can have slope fields.
- In the context of a slope field, an initial condition $y(x_0) = y_0$ simply specifies that the graph of the solution must pass through (x_0, y_0).

Example 10

Sketch the slope field of

$$\frac{dy}{dx} = -y^2$$

and compare it to the general solution

$$y = \frac{1}{x + C}.$$

Example 11

Sketch the slope field of

$$\frac{dy}{dx} = 2 - xy$$

and several solution curves. What can you conclude about the behavior of the solutions as $x \to \pm\infty$?

Example 12

Sketch the slope field of

$$\frac{dy}{dx} = x - y$$

and several solution curves. What can you conclude about the behavior of the solutions as $x \to \pm\infty$?