Separable ODEs

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Calculus II

Introduction

The first class of ODEs we will study are the (first order) *separable* ODEs.

We will soon see that up to our ability to compute antiderivatives, every separable ODE can be solved, at least implicitly.

Applications of separable equations include *mixing problems* and *Newton's law of cooling*.

Separable ODEs

Definition

A (first order) ODE is called *separable* if it has one of the (equivalant) forms

$$\frac{dy}{dx} = f(x)g(y) \quad \text{or} \quad \frac{dy}{dx} = \frac{f(x)}{h(y)}.$$

Remark. To see that these are equivalent simply set

$$g(y) = \frac{1}{h(y)}.$$

Example. The ODE

$$\frac{dy}{dx} = \frac{3x^2 - 4}{2y}$$

is separable.

Solving Separable ODEs

In principle, it is always possible to solve a separable ODE.

If we treat the derivative as a fraction we have

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \;\Rightarrow\; g(y)\,dy = f(x)\,dx$$
$$\Rightarrow \int g(y)\,dy = \int f(x)\,dx.$$

It remains to:

1. Compute both indefinite integrals.
2. Algebraically solve for y in terms of x.

Remark. Although this procedure is totally formal, it is easily justified.

Example 1

Solve the ODE $y' = -y^2$.

Solution. We write $y' = \frac{dy}{dx}$ and proceed as outlined above:

$$\frac{dy}{dx} = -y^2 \;\underset{y\neq 0}{\Longrightarrow}\; \frac{-dy}{y^2} = dx \;\Rightarrow\; \int \frac{-dy}{y^2} = \int dx$$

$$\Rightarrow\; \frac{1}{y} = x + C \;\Rightarrow\; y = \frac{1}{x+C}.$$

So the complete list of solutions is

$$\boxed{y = \frac{1}{x+C} \quad \text{and} \quad y = 0}.$$

□

Example 2

Solve the natural growth equation $\frac{dy}{dx} = ky$.

Solution. Since this is a separable equation, if $y \neq 0$ we have

$$\frac{dy}{y} = k\,dx \;\Rightarrow\; \ln|y| = kx + C_0 \;\Rightarrow\; |y| = e^{kx+C_0} = e^{C_0}e^{kx}$$

$$\Rightarrow\; y = \pm e^{C_0}e^{kx}.$$

Since C_0 is arbitrary, we can replace $\pm e^{C_0}$ with $C \neq 0$. But $y = 0$ is also a solution, so the general solution is

$$\boxed{y = Ce^{kx}}.$$

□

Example 3

Solve the ODE $\dfrac{dy}{dx} = 3 + 2x + 3y + 2xy$.

Solution. To see that this is separable, we must factor the RHS:

$$\frac{dy}{dx} = (3+3y) + (2x+2xy) = 3(1+y) + 2x(1+y)$$

$$= (3+2x)(1+y) \underset{y \neq -1}{\Longrightarrow} \int \frac{dy}{1+y} = \int 3 + 2x \, dx$$

$$\Rightarrow \ln|1+y| = 3x + x^2 + C \Rightarrow |1+y| = e^{x^2+3x+C} = e^C e^{x^2+3x}$$

$$\Rightarrow 1 + y = \pm e^C e^{x^2+3x} = Ce^{x^2+3x} \Rightarrow \boxed{y = -1 + Ce^{x^2+3x}}.$$

Note that this formula includes the "missing" solution $y = -1$. □

Example 4

Solve the IVP

$$\frac{dy}{dx} = \frac{y\cos x}{1+y^2}, \quad y(0) = 1.$$

Solution. First we find the general solution to the (separable) ODE:

$$\frac{dy}{dx} = \frac{y\cos x}{1+y^2} \underset{y\neq 0}{\Longrightarrow} \int \frac{1+y^2}{y}\,dy = \int \cos x\,dx$$

$$\Rightarrow \int \frac{1}{y} + y\,dy = \sin x + C$$

$$\Rightarrow \ln|y| + \frac{y^2}{2} = \sin x + C.$$

To solve for C, we plug in $x = 0$, $y = 1$ to get

$$\ln|1| + \frac{1}{2} = \sin 0 + C \;\Rightarrow\; C = \frac{1}{2}.$$

Because we can't reasonably solve for y, we leave the solution in *implicit form*:

$$\boxed{\ln|y| + \frac{y^2}{2} = \sin x + \frac{1}{2}}.$$

□

Remark. Whenever possible, we will prefer to express our solutions *explicitly* in the form $y = f(x)$.

Example 5

Solve the IVP

$$\frac{dy}{dx} = \frac{3x^2 - 4}{2y}, \quad y(0) = -2.$$

Solution. First we find the general solution of the ODE:

$$\frac{dy}{dx} = \frac{3x^2 - 4}{2y} \;\Rightarrow\; \int 2y\,dy = \int 3x^2 - 4\,dx$$

$$\Rightarrow\; y^2 = x^3 - 4x + C.$$

Before "burying" the constant C, we solve for it using the initial condition $x = 0$, $y = -2$:

$$(-2)^2 = 0^3 - 4 \cdot 0 + C \;\Rightarrow\; C = 4.$$

So the solution is given by

$$y^2 = x^3 - 4x + 4 \;\;\Rightarrow\;\; y = \pm\sqrt{x^3 - 4x + 4}.$$

Since we need $y(0) = -2 < 0$, we must choose the negative sign.

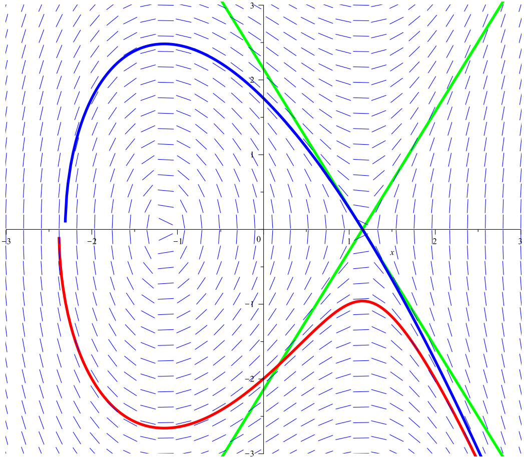
$$\boxed{y = -\sqrt{x^3 - 4x + 4}}.$$

□

Remark. It is interesting to note that the general solution is

$$y = \pm\sqrt{x^3 - 4x + C},$$

whose domain *depends on* C. This can be seen in the slope field where $y = 0$ and dy/dx becomes infinite (vertical).



Slope field for

$$\frac{dy}{dx} = \frac{3x^2 - 4}{2y}$$

illustrating the solution to the IVP with $y(0) = -2$ (red) and the "double tangent" at the point where both the numerator and denominator vanish simultaneously.