Applications of Separable ODEs

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Calculus II

Mixing Problems

In a *mixing problem* a certain substance X is dissolved in a fluid that is contained in a "tank."

Fluid containing (more or less of) substance X enters the tank at a certain rate.

After being thoroughly mixed, the fluid drains out of the tank at a certain rate as well.

If $V(t)$ denotes the total amount of substance X in the tank at time t, we arrive at a differential equation of the form

$$\frac{dV}{dt} = \begin{pmatrix} \text{rate at which } X \\ \text{enters the tank} \end{pmatrix} - \begin{pmatrix} \text{rate at which } X \\ \text{leaves the tank} \end{pmatrix} = R_{in} - R_{out}.$$

Example 1

The concentration of CO_2 in a room with a volume of 180 cubic meters is initially 0.15%. Fresh air containing only 0.05% CO_2 enters the room at a rate of 2 cubic meters per minute. The air is mixed well by a fan in the room and then leaves the room at the same rate. Find the concentration of CO_2 in the air in the room as a function of time. What happens to the concentration in the long run?

Solution. The first thing we do is introduce the relevant variables and units. Let:

$$t = \text{time (minutes)}, \quad V = \text{amount of } CO_2 \text{ in the room } (\text{m}^3),$$

$$R_{\text{in}} = \text{rate at which } CO_2 \text{ enters} = \frac{2 \cdot 0.0005\,\text{m}^3}{\text{min.}} = 0.001\frac{\text{m}^3}{\text{min}}$$

$$R_{\text{out}} = \text{rate at which } CO_2 \text{ leaves} = \frac{V}{180} \cdot \frac{2\,\text{m}^3}{\text{min.}} = \frac{V}{90}\,\frac{\text{m}^3}{\text{min.}}$$

We therefore have

$$\frac{dV}{dt} = R_{\text{in}} - R_{\text{out}} = 0.001 - \frac{V}{90} = \frac{1}{1000} - \frac{V}{90} = \frac{9 - 100V}{9000}$$

together with the initial condition

$$V(0) = 0.0015 \cdot 180 = \frac{27}{100}.$$

The general solution is given by

$$\int \frac{dV}{9 - 100V} = \int \frac{1}{9000} dt \;\Rightarrow\; -\frac{1}{100} \ln|9 - 100V| = \frac{t}{9000} + C$$

$$\Rightarrow\; \ln|9 - 100V| = -\frac{t}{90} + C.$$

The initial condition tells us that

$$\ln\left|9 - 100 \cdot \frac{27}{100}\right| = 0 + C \;\Rightarrow\; C = \ln 18.$$

Therefore

$$
\begin{aligned}
\ln|9-100V| &= -\frac{t}{90}+\ln 18\\
&\Rightarrow\ |9-100V| = e^{-t/90+\ln 18} = 18e^{-t/90}\\
&\Rightarrow\ 9-100V = \pm 18e^{-t/90}\\
&\Rightarrow\ V = \frac{9\pm 18e^{-t/90}}{100}.
\end{aligned}
$$

In order to have $V(0)>0$, we must choose the positive sign:

$$V=\frac{9+18e^{-t/90}}{100}.$$

The concentration is then

$$\frac{V}{180}\cdot 100\,\% = \frac{1+2e^{-t/90}}{20}\,\% = \boxed{0.05+0.1e^{-t/90}\,\%}.$$

Note that

$$\lim_{t\to\infty} 0.05 + 0.1e^{-t/90} = 0.05,$$

which tells us that in the long run the concentration of CO_2 in the room will (rapidly) approach the concentration of CO_2 in the incoming air. □

Remark. It is tempting to instead let C denote the concentration of CO_2 (% vol.) and set

$$\frac{dC}{dt} = 2 \cdot 0.05 - 2C,$$

but this *cannot* be correct: the units of the LHS are %/min., while the units of the RHS are $(\mathrm{m}^3/\mathrm{min})\%$.

To get the correct units we need to divide the RHS by a volume.

It turns out that the volume of the tank works.

That is, the correct ODE for C is

$$\frac{dC}{dt} = \frac{2 \cdot 0.05 - 2C}{180}.$$

Moral. To correctly set up a mixing problem, the dependent variable should be the *amount* of the solute, *not* its percentage concentration.

Newton's Law of Cooling

Suppose we have a heated object H which radiates heat to its surroundings, thereby changing its temperature.

Newton's law of cooling states that the time rate of change of the temperature of H is proportional to the difference between H's temperature and the temperature of its surroundings (the *ambient temperature*).

That is, if $u(t)$ denotes the temperature of H at time t, and $A(t)$ denotes the ambient temperature at time t, then

$$\frac{du}{dt} = k(A(t) - u(t))$$

for some constant k.

Since our experience tells us that warm objects tend to cool off, we should have $k > 0$, so that $\frac{du}{dt}$ is negative when $u(t) > A(t)$.

Note that $\frac{du}{dt}$ is large (and negative) when $u(t)$ is much larger than $A(t)$, and that $\frac{du}{dt}$ is small when $u(t)$ is close to $A(t)$.

This tells us that a heated object cools rapidly, but its temperature approaches the ambient temperature in the long run.

Likewise, Newton's law of cooling implies that a chilled object will warm rapidly.

Constant Ambient Temperature

If the ambient temperature is *constant* (e.g. a coffee cup in a room or a turkey in an oven), then

$$\frac{du}{dt} = k(A - u)$$

is separable, and easily solved.

Example 2

Coffee at 150°F is brought into an air-conditioned room kept at $70^\circ F$. After 15 minutes the coffee has cooled to 100°F. Determine the temperature of the coffee at any time after it was brought into the room.

Solution

Assuming the coffee cools according to Newton's law of cooling, its temperature u at time t minutes after it was brought into the room satisfies

$$\frac{du}{dt} = k(70 - u)$$

for some unknown $k > 0$.

We are also given

$$u(0) = 150 \quad \text{and} \quad u(15) = 100.$$

Separating variables in the ODE we obtain

$$\int \frac{du}{u - 70} = \int -k\,dt \;\Rightarrow\; \ln|u - 70| = -kt + C.$$

We can solve for k and C more easily if we impose our boundary conditions now (rather than after we've solved for u).

Setting $t = 0$, $u = 150$ yields

$$\ln|150 - 70| = -k \cdot 0 + C \quad \Rightarrow \quad C = \ln 80.$$

Setting $t = 15$, $u = 100$ yields

$$\ln|100 - 70| = -15k + \ln 80 \quad \Rightarrow \quad k = \frac{\ln 80 - \ln 30}{15} = \frac{\ln(8/3)}{15}.$$

Thus

$$\ln|u - 70| = -\frac{\ln(8/3)}{15}t + \ln 80 \quad \Rightarrow \quad |u - 70| = \exp\left(-\frac{\ln(8/3)}{15}t + \ln 80\right)$$

$$\Rightarrow \quad u - 70 = \pm e^{\ln 80} \exp\left(-\frac{\ln(8/3)}{15} t\right)$$

$$\Rightarrow \quad \boxed{u = 70 + 80 \exp\left(-\frac{\ln(8/3)}{15} t\right)},$$

our choice of sign being dictated by the fact that we need $u(0) = 150$. □

Remark. The value of k is

$$k = \frac{\ln(8/3)}{15} \approx 0.0653886\ldots$$

Remark

In the preceding example, one can also estimate k from the data points

$$u(0) = 150 \quad \text{and} \quad u(15) = 100$$

as follows:

$$k(70-u) = \frac{du}{dt} \approx \frac{\Delta u}{\Delta t} = \frac{-50}{15} = -\frac{10}{3} \;\Rightarrow\; k \approx \frac{10}{3(u-70)}.$$

But which u-value should we use?

The slope field suggests that $\frac{du}{dt}$ will more closely approximate $\frac{\Delta u}{\Delta t}$ at the midpoint $u = \frac{150+100}{2} = \frac{250}{2} = 125$ than at either data point.

So we expect to have

$$k \approx \frac{10}{3(125-70)} = \frac{10}{3 \cdot 55} = \frac{2}{33} = 0.06060606\ldots$$

This is reasonably close to the exact value

$$k = 0.0653886\ldots$$

found above.