First Order Linear ODEs

Ryan C. Daileda

Trinity University

Calculus II

Introduction

So far we have discussed how to solve two types of ODEs:

- Antiderivatives: $y' = f(x)$
- Separable ODEs: $\dfrac{dy}{dx} = \dfrac{f(x)}{g(y)}$

In both cases, we manipulated the equation algebraically (if necessary), then integrated both sides. This will remain our basic approach.

We can solve separable ODEs because of the chain rule (essentially). The product rule allows us to treat *linear ODEs*.

Motivation

Example 1

Solve the first order ODE

$$x\frac{dy}{dx} + y = \frac{1}{x^2+1}.$$

Solution. After staring at this for a while, we notice that the LHS is just the result of the product rule for derivatives:

$$x\frac{dy}{dx} + y = \frac{d}{dx}(xy).$$

So we must solve

$$\frac{d}{dx}(xy) = \frac{1}{x^2+1} \;\Rightarrow\; xy = \int \frac{dx}{x^2+1} = \arctan x + C$$

$$\Rightarrow\; \boxed{y = \frac{\arctan x + C}{x}}.$$

Example 2

Solve the first order ODE

$$\frac{dy}{dx} = x - y.$$

Solution. First we rewrite the equation as

$$\frac{dy}{dx} + y = x.$$

We can't use the product rule right away, so we multiply both side by $m(x) = e^x$:

$$e^x \frac{dy}{dx} + e^x y = xe^x \;\Rightarrow\; \frac{d}{dx}(e^x y) = xe^x \;\Rightarrow\; e^x y = \int xe^x\, dx$$

$$\Rightarrow\; e^x y = xe^x - e^x + C \;\Rightarrow\; \boxed{y = x - 1 + Ce^{-x}}.$$

$\square$

Integrating Factors

The function $m(x) = e^x$ in the previous example is called an *integrating factor* (*factor* because we multiply by $m(x)$; *integrating* because this enables us to directly integrate).

Question. In which situations will something similar also work?

Definition

A (first order) linear ODE has the form

$$\frac{dy}{dx} + p(x)y = q(x).$$

Remark. We will define the term *linear* more carefully later. For now it is simply an adjective describing the shape of the ODE.

In principle, it is always possible to solve a linear ODE. Starting with

$$y' + py = \frac{dy}{dx} + py = q,$$

we set

$$m(x) = \exp\left(\int p(x)\,dx\right) = e^{\int p\,dx}.$$

Notice that $m' = \frac{dm}{dx} = p(x)e^{\int p\,dx} = pm$. So if we multiply both sides of the ODE by m we find that

$$(my)' = m\frac{dy}{dx} + \underbrace{mp}_{m'}\,y = mq \;\Rightarrow\; my = \int mq\,dx$$

$$\Rightarrow\; y = \frac{1}{m}\int mq\,dx.$$

To obtain an explicit form of the solution, It only remains to antidifferentiate (p and) mq.

Examples

Example 3

Solve the ODE $\dfrac{dy}{dx} = x + y$.

Solution. First we put the ODE in "standard" form,

$$\frac{dy}{dx} - y = x \quad \left(\frac{dy}{dx} + py = q\right) \;\Rightarrow\; p(x) = -1.$$

Next we compute the integrating factor:

$$m(x) = \exp\left(\int p(x)\,dx\right) = e^{\int -1\,dx} = e^{-x}.$$

Now multiply the ODE through by $m(x) = e^{-x}$, invoke the product rule, and integrate:

$$e^{-x}\frac{dy}{dx} - e^{-x}y = xe^{-x} \;\Rightarrow\; \frac{d}{dx}(e^{-x}y) = xe^{-x}$$

$$\Rightarrow\; e^{-x}y = \int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

$$\Rightarrow\; \boxed{y = -x - 1 + Ce^{x}}.$$

□

Remarks.

- You *must* write the ODE as

$$y' + py = q \quad \text{or} \quad \frac{dy}{dx} + py = q$$

  in order to correctly compute the integrating factor.
- The solution to a linear ODE always has the form

$$y = a(x) + C \cdot b(x).$$

Example 4

Solve the ODE $\dfrac{dy}{dx} + (\cot x)y = x \csc x$.

Solution. This is already in the correct form, so we compute

$$m(x) = \exp\left(\int \cot x \, dx\right) = \exp\left(\int \frac{\cos x}{\sin x} \, dx\right)$$
$$= \exp(\ln \sin x) = \sin x.$$

Now multiply the ODE through by $m(x)$:

$$(\sin x)\frac{dy}{dx} + (\cos x)y = x \;\Rightarrow\; \frac{d}{dx}(y \sin x) = x$$
$$\Rightarrow\; y \sin x = \int x \, dx = \frac{x^2}{2} + C \;\Rightarrow\; \boxed{y = \frac{x^2}{2 \sin x} + C \csc x}.$$

$\square$

Example 5

Solve $\dfrac{y'}{x} + y = 1$.

Solution. We have:

$$y' + xy = x \;\Rightarrow\; m(x) = e^{\int x\,dx} = e^{x^2/2}.$$

Now multiply the ODE by $m(x)$:

$$(e^{x^2/2}y)' = e^{x^2/2}y' + xe^{x^2/2}y = xe^{x^2/2}$$

$$\Rightarrow\; e^{x^2/2}y = \int xe^{x^2/2}\,dx = e^{x^2/2} + C$$

$$\Rightarrow\; \boxed{y = 1 + Ce^{-x^2/2}}.$$

□

Example 6

Solve the ODE $\frac{dy}{dx} + xy = 1$.

Solution. Since the LHS is the same as in the preceding example, we can use the same integrating factor:

$$m(x) = e^{x^2/2}.$$

So the ODE becomes

$$\frac{d}{dx}(e^{x^2/2}y) = e^{x^2/2}\frac{dy}{dx} + xe^{e^x/2}y = e^{x^2/2}$$

$$\Rightarrow \quad e^{x^2/2}y = \int e^{x^2/2}\,dx \quad \Rightarrow \quad \boxed{y = e^{-x^2/2}\int e^{x^2/2}\,dx}.$$

$\square$

Remark. We leave the solution in this form since $e^{x^2/2}$ does not have an elementary antiderivative.

Example 7

Solve the ODE $x^2y' + 3xy = \dfrac{\sin x}{x}$.

Solution. First we obtain the "standard" form

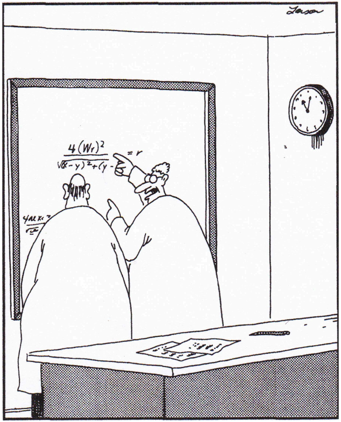
$$y' + \frac{3}{x}y = \frac{\sin x}{x^3} \quad \Rightarrow \quad m(x) = \exp\left(\int \frac{3}{x}\,dx\right) = e^{3\ln x} = x^3.$$

Multiplying the ODE by $m(x)$ yields

$$(x^3y)' = x^3y' + 3x^2y = \sin x \quad \Rightarrow \quad x^3y = \int \sin x\,dx = -\cos x + C$$

$$\Rightarrow \quad \boxed{y = \frac{C - \cos x}{x^3}}.$$

□



"Yes, yes, I *know* that, Sidney ... *every*body knows *that*! ... But look: Four wrongs *squared*, minus two wrongs to the fourth power, divided by this formula, *do* make a right."