Calculus II: Day 1

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Calculus II

Introduction

In Calculus II we apply and extend the ideas encountered in Calculus I. Specifically, we will study:

Techniques of Integration. Methods for finding antiderivatives of functions.

- General techniques: substitution and integration by parts.
- Specific techniques: integration of trigonometric polynomials, trigonometric substitution, integration of rational functions, etc.
- Improper integrals: integration of functions with discontinuities or unbounded domains.

Ordinary Differential Equations (ODEs). Methods for solving certain classes of ODEs, e.g.

$$y' + xy = x^3 \quad \text{or} \quad y'' + 2y' + 3y = \sin x.$$

- First order ODEs: linear ODEs, separable ODEs, autonomous ODEs.
- Second order ODEs: fundamental solutions to linear ODEs, equations with constant coefficients, homogeneous and inhomogeneous equations.

Infinite Series. What does it mean to "add" and infinite number of quantities, e.g. what does

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

mean?

- Convergence tests: conditional and absolute convergence, geometric series, the integral test, root and ratio tests, comparison tests, etc.
- Power series: radius of convergence, analytic properties.
- Series representations of functions: Taylor series expansions.

The Definite Integral

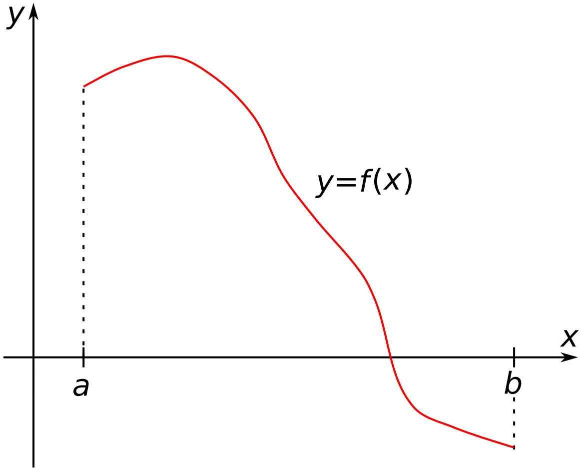
Given a (continuous) function $f(x)$ on an interval $[a, b]$:

$$\int_a^b f(x)\,dx = \underbrace{\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i}_{\text{Riemann sum}},$$

where:

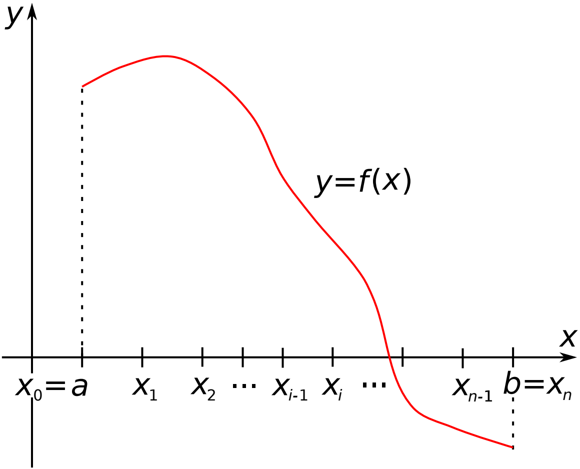
- $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ is a *partition* (subdivision) of $[a, b]$;
- x_i^* is a *sample point* in the subinterval $[x_{i-1}, x_i]$;
- $\Delta x_i = x_i - x_{i-1}$ and $\Delta x = \max\limits_{1 \le i \le n} \Delta x_i$.

Graphical Interpretation



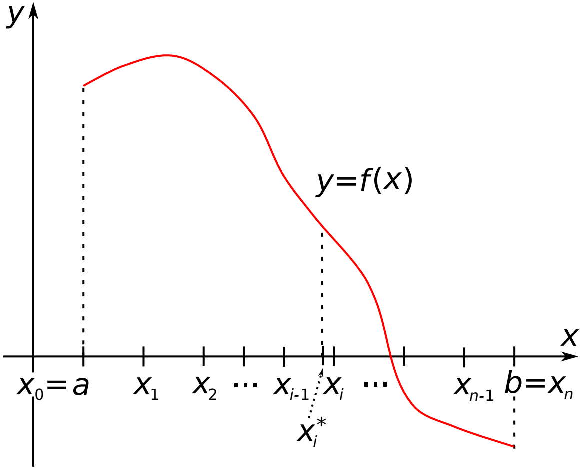
A function $f(x)$ on $[a, b]$.

Graphical Interpretation



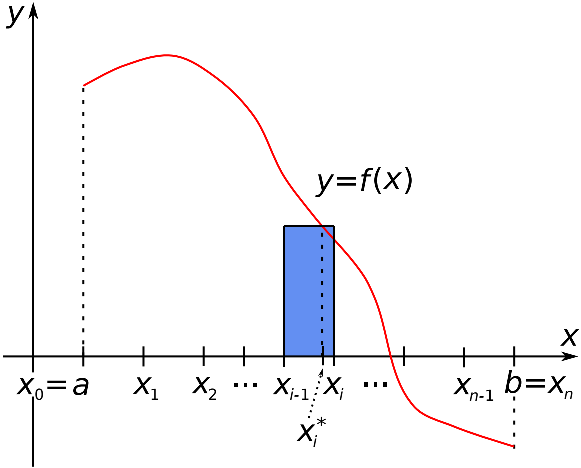
A partition (subdivision) of $[a, b]$.

Graphical Interpretation



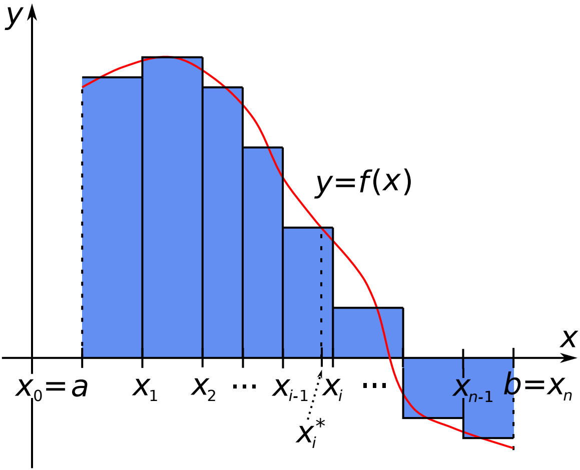
A sample point x_i^* in the subinterval $[x_{i-1}, x_i]$.

Graphical Interpretation



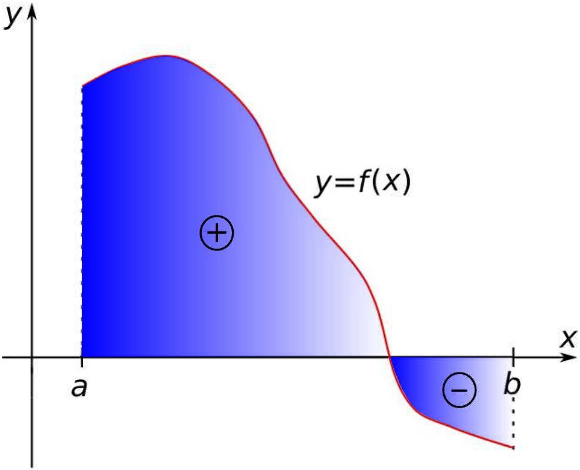
The term $f(x_i^*)\Delta x_i$ represents the (signed) area of the rectangle.

Graphical Interpretation



The Riemann sum $\sum_i f(x_i^*)\Delta x_i$ represents the total (signed) area of the rectangles.

Graphical Interpretation



In the limit, the integral $\int_a^b f(x)\,dx$ represents the *exact* (signed) area between the graph and the x-axis.

Remarks

The graphical interpretation is the easiest way to understand $\int_a^b f(x)\,dx$ for a "generic" $f(x)$.

In applications, however, the Riemann sums $\sum_i f(x_i^*)\Delta x_i$, and hence the integral, have other interpretations.

Other quantities computed with integrals via Riemann sums include:

- Displacement/distance (from velocity)
- Area/volume of surfaces/regions of revolution
- Probabilities
- Double, triple, line and surface integrals (in Calc. III)
- Mass, center of mass, electric charge, electric flux, etc.

Computing Definite Integrals

Moral. Because of their wide applicability, it is important to be able to compute (definite) integrals.

The *Fundamental Theorem of Calculus* (FTOC) tells us how to do this with antiderivatives.

Theorem 1 (FTOC, Part II)

If $f(x)$ is continuous and $F'(x) = f(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b.$$

Remarks

Finding antiderivatives is *not* the only way to evaluate definite integrals, however this is the technique we will focus on.

Even though differentiation is straightforward, finding (explicit) antiderivatives can be difficult or even impossible.

For example, the function

$$f(x) = \frac{x^4 - x^3 + 5}{x^2 - x + 2}$$

can easily be differentiated with the quotient and power rules, yet its antiderivative is

$$F(x) = \frac{1}{3}x^3 - 2x - \ln(x^2 - x + 2) + \frac{16}{\sqrt{7}} \arctan\left(\frac{2x - 1}{\sqrt{7}}\right) + C.$$

Notation and Terminology

Motivated by FTOC, we define the *indefinite integral* of $f(x)$ to be

$$\int f(x)\,dx = \text{the general antiderivative of } f(x).$$

Be aware that:

- $\int f(x)\,dx$ is a (set of) *function*(s);
- $\int_a^b f(x)\,dx$ is a *number*.

The indefinite and definite integrals are connected through FTOC.

When there is no risk of confusion, we will use the terms "integral" and "antiderivative" interchangeably.

Basic Rules of Integration

If $F'(x) = f(x)$, then

$$\int f(x)\,dx = F(x) + C,$$

where C is an arbitrary constant (don't forget it!).

The (indefinite) integral is linear:

$$\int af(x) + bg(x)\,dx = a\int f(x)\,dx + b\int g(x)\,dx.$$

For $n \neq -1$ the power rule yields

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

We also have

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C.$$

For the exponential and trigonometric functions:

$$\int e^x\,dx = e^x + C,$$
$$\int \cos x\,dx = \sin x + C,$$
$$\int \sin x\,dx = -\cos x + C.$$

And we have the *substitution rule*

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Examples

Example 1

Compute $\int x^3 + 2x + 5\,dx$.

Solution. Using linearity and the power rule we immediately have

$$\int x^3 + 2x + 5\,dx = \frac{x^4}{4} + \frac{2x^2}{2} + 5x + C$$

$$= \boxed{\frac{x^4}{4} + x^2 + 5x + C}.$$

□

Example 2

Compute $\int \frac{x^{3/2}+1}{2x}\,dx$.

Solution. We simplify and use the power rule:

$$\int \frac{x^{3/2}+1}{2x}\,dx = \int \frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1}\,dx$$

$$= \frac{1}{2}\cdot\frac{x^{3/2}}{3/2} + \frac{1}{2}\ln|x| + C$$

$$= \boxed{\frac{x^{3/2}}{3} + \frac{1}{2}\ln|x| + C}.$$

□

Example 3

Compute $\int x(3x^2+7)^{50}\,dx$.

Solution. We perform the substitution

$$u = 3x^2 + 7 \;\Rightarrow\; du = 6x\,dx \;\Rightarrow\; x\,dx = \frac{1}{6}\,du.$$

Thus

$$\begin{aligned}\int x(3x^2+7)^{50}\,dx &= \frac{1}{6}\int u^{50}\,du \\ &= \frac{1}{6}\cdot\frac{u^{51}}{51} + C \\ &= \boxed{\frac{(3x^2+7)^{51}}{306} + C}.\end{aligned}$$

□

Example 4

Compute $\displaystyle\int \frac{x^2}{x^3+1}\,dx$.

Solution. Again, we substitute:

$$u = x^3 + 1 \;\Rightarrow\; du = 3x^2\,dx \;\Rightarrow\; x^2\,dx = \frac{1}{3}\,du.$$

Thus

$$\begin{aligned}\int \frac{x^2}{x^3+1}\,dx &= \frac{1}{3}\int \frac{du}{u}\\ &= \frac{1}{3}\ln|u| + C\\ &= \boxed{\frac{1}{3}\ln\left|x^3+1\right| + C}.\end{aligned}$$

□

Example 5

Compute $\int x\sqrt{x+1}\,dx$.

Solution. We substitute

$$u = x + 1 \;\Rightarrow\; \begin{array}{c} x = u - 1, \\ du = dx. \end{array}$$

Then we have

$$\begin{aligned} \int x\sqrt{x+1}\,dx &= \int (u-1)\sqrt{u}\,du = \int u^{3/2} - u^{1/2}\,du \\ &= \frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2} + C \\ &= \boxed{\frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C}. \end{aligned}$$

□

Example 6

Compute $\int_0^{\pi} e^{\cos x} \sin x \, dx$.

Solution. We substitute

$$u = \cos x \quad \Rightarrow \quad du = -\sin x \, dx,$$

and change the limits of integration to obtain

$$\int_0^{\pi} e^{\cos x} \sin x \, dx = -\int_{\cos 0}^{\cos \pi} e^u \, du = \int_{-1}^{1} e^u \, du$$

$$= e^u \Big|_{-1}^{1} = \boxed{e - \frac{1}{e}}.$$

□

Example 7

Compute $\int_1^2 \frac{x}{x^4+1}\,dx$.

Solution. Notice that

$$\frac{x}{x^4+1} = \frac{x}{(x^2)^2+1}.$$

So we substitute

$$u = x^2 \;\Rightarrow\; du = 2x\,dx \;\Rightarrow\; x\,dx = \frac{1}{2}\,du.$$

Thus

$$\int_1^2 \frac{x}{x^4+1}\,dx = \frac{1}{2}\int_1^4 \frac{du}{u^2+1} = \frac{1}{2}\arctan u\Big|_1^4$$

$$= \frac{1}{2}\arctan 4 - \frac{1}{2}\arctan 1$$

$$= \boxed{-\frac{\pi}{8} + \frac{1}{2}\arctan 4}.$$

□

Remark. You can *always* (in principle) check if an antiderivative computation is correct simply by differentiating your answer!

Example 8

Verify that $\int \ln x\, dx = x \ln x - x + C$.

Solution. We differentiate the given result:

$$\frac{d}{dx}(x \ln x - x + C) = 1 \cdot \ln x + x \cdot \frac{1}{x} - 1$$
$$= \ln x + 1 - 1 = \ln x,$$

which is what we needed to show. □

Example 9

Verify that $\displaystyle\int \frac{1}{x^2+x-2}\,dx = \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C.$

Solution. We simplify the RHS then differentiate:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C\right) &= \frac{1}{3}\frac{d}{dx}\left(\ln|x-1| - \ln|x+2|\right) \\
&= \frac{1}{3}\left(\frac{1}{x-1} - \frac{1}{x+2}\right) \\
&= \frac{1}{3}\cdot\frac{(x+2)-(x-1)}{(x-1)(x+2)} = \frac{1}{x^2+x-2},
\end{aligned}$$

which is what we needed to show. □