Volumes by Cross Sections

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Calculus II

Introduction

Although it can be "generically" interpreted as a signed area, the true utility of the definite integral lies in its connection to Riemann sums.

Over the course of the next few lectures we will consider various applications of the definite integral in this context.

For instance, today we will see how volume can be computed from cross-sectional area using an integral.

We will see additional applications once we have more advanced integration techniques.

Cross Sections and Volumes

Suppose that S is a solid object and that $A(x)$ gives the cross-sectional area of S, perpendicular to the x-axis.

Diagram/See Maple.

A Riemann sum argument can be used to show that

$$\boxed{\mathrm{Vol}(S) = \int_a^b A(x)\,dx,}$$

where the cross sections run over the interval $a \leq x \leq b$.

Examples

Example 1

The base of a solid is a disk of radius r. Cross sections perpendicular to the base are squares. Compute the volume of the solid.

Solution. See Maple for an image of the solid.

Center the disk at the origin.

(diagram)

The cross section at position x is a square with side length

$$\ell(x) = 2\sqrt{r^2 - x^2}.$$

Thus

$$A(x) = \ell(x)^2 = \left(2\sqrt{r^2 - x^2}\right)^2 = 4(r^2 - x^2),$$

for $-r \leq x \leq r$.

Hence

$$\begin{aligned}
\text{Vol.} &= \int_{-r}^{r} 4(r^2 - x^2)\, dx \\
&= 4\left(r^2 x - \frac{x^3}{3}\right)\Bigg|_{x=-r}^{x=r} \\
&= 8 \cdot \frac{2r^3}{3} \\
&= \boxed{\frac{16r^3}{3}}.
\end{aligned}$$

□

Example 2

The region bounded by the curves $y = 1/x$, $y = 0$, $x = 1$ and $x = 2$ is rotated about the x-axis to produce a solid. Find its volume.

Solution. See Maple for an image of the solid.

The cross sections perpendicular to the x-axis are disks with radii

$$r(x) = y = 1/x.$$

(diagram)

Thus

$$A(x) = \pi r(x)^2 = \frac{\pi}{x^2},$$

for $1 \le x \le 2$.

Hence

$$\begin{aligned}
\text{Vol.} &= \pi \int_1^2 \frac{1}{x^2}\,dx = \pi \int_1^2 x^{-2}\,dx \\
&= \pi \left(\frac{x^{-1}}{-1}\bigg|_1^2 \right) = \pi \left(1 - \frac{1}{2}\right) \\
&= \boxed{\frac{\pi}{2}}.
\end{aligned}$$

$\square$

Example 3

The region bounded by $y = \sqrt{x}$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

Solution. See Maple for an image of the solid.

Cross sections perpendicular to the x-axis are annuli ("washers") with outer radius $R(x) = \sqrt{x}$ and inner radius $r(x) = x^2$.

(diagram)

Hence

$$A(x) = \pi R(x)^2 - \pi r(x)^2 = \pi \left(x - x^4\right),$$

for $0 \le x \le 1$.

Thus

$$\text{Vol.} = \pi \int_0^1 x - x^4 \, dx = \pi \left(\frac{x^2}{2} - \frac{x^5}{5} \bigg|_0^1 \right)$$

$$= \pi \left(\frac{1}{2} - \frac{1}{5} \right) = \boxed{\frac{3\pi}{10}}.$$

□

Example 4

Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = \sqrt{x}$ about the line $y = 1$.

Solution. See Maple for an image of the solid.

The cross sections perpendicular to the x-axis are again annuli, but this time the outer radius is $R(x) = 1 - x$, while the inner radius is $r(x) = 1 - \sqrt{x}$.

(diagram)

Thus

$$\begin{aligned} A(x) &= \pi R(x)^2 - \pi r(x)^2 = \pi((1-x)^2 - (1-\sqrt{x})^2) \\ &= \pi(x^2 - 3x + 2\sqrt{x}), \end{aligned}$$

for $0 \leq x \leq 1$.

Therefore

$$\text{Vol.} = \pi \int_0^1 x^2 - 3x + 2\sqrt{x}\, dx = \pi \left(\frac{x^3}{3} - \frac{3x^2}{2} + \frac{4}{3}x^{3/2} \Big|_0^1 \right)$$

$$= \pi \left(\frac{1}{3} - \frac{3}{2} + \frac{4}{3} \right) = \boxed{\frac{\pi}{6}}.$$

$\square$

Example 5

Find the volume of the solid obtained by rotating the region bounded by $x = y^2$ and $x = 2y$ about the y-axis.

Solution. We take cross sections perpendicular to the y-axis instead.

The cross section at height y is an annulus with inner radius $r(y) = y^2$ and outer radius $R(y) = 2y$.

(diagram)

Thus

$$A(y) = \pi R(y)^2 - \pi r(y)^2 = \pi(4y - y^4),$$

for $0 \leq y \leq 2$.

Therefore

$$\begin{aligned} \text{Vol.} &= \pi \int_0^2 4y - y^4\, dy \\ &= \pi \left(2y^2 - \frac{y^5}{5} \bigg|_0^2 \right) \\ &= \pi \left(8 - \frac{32}{5} \right) = \boxed{\frac{8\pi}{5}}. \end{aligned}$$

□