Volumes by Cylindrical Shells

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Calculus II

Introduction

For solids of revolution, the method of computing volumes by cross sections isn't always feasible.

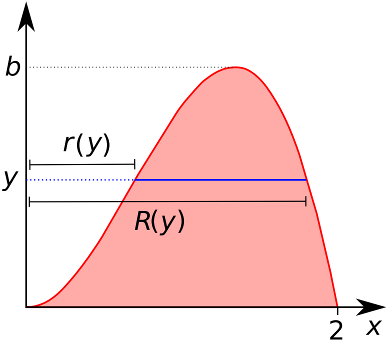
However, by taking advantage of the rotational symmetry of such solids, we can derive an entirely different means of computing volume.

This leads to the *method of cylindrical shells*, which can be very useful in certain situations.

Before developing this technique in general, we look at an example to motivate it.

Motivation

Suppose we are asked to find the volume of the solid obtained by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y-axis.



Cross sections perpendicular to the y-axis are washers, with inner and outer radii as shown.

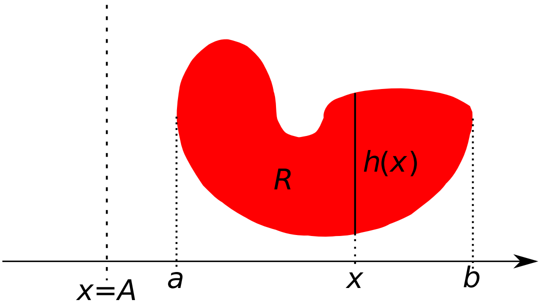
Uh oh...

We are immediately faced with two problems:

- To find $r(y)$ and $R(y)$ (which are x-values) we need to solve $y = 2x^2 - x^3$ for x, which is no easy task.
- To find b we need to find the maximum value of $y = 2x^2 - x^3$, which is an optimization problem.

In other words, computing the volume using cross sections is going to be extremely difficult. Is there a better way?

Suppose we are given a region R in the xy-plane that lies between the vertical lines $x = a$ and $x = b$, and that for each $x \in [a, b]$ we know the height $h(x)$ of the vertical cross section of R.



Create a solid S by rotating R about the vertical axis $x = A$.

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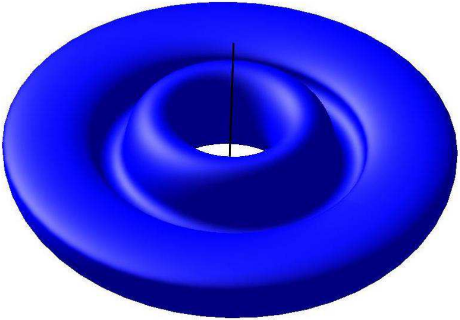
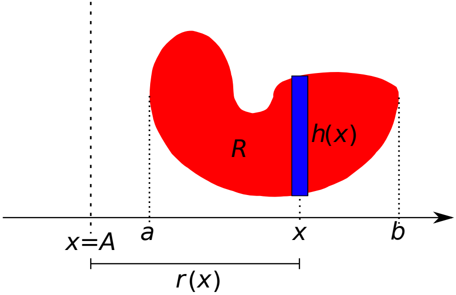


Create a solid S by rotating R about the vertical axis $x = A$.

Cylindrical Shells

To compute the volume of S, we first take thin slices of R of width Δx.



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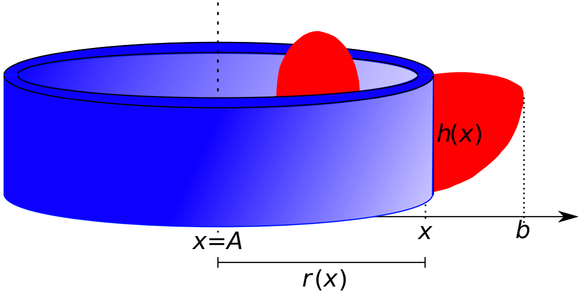


When we rotate around $x = A$, each slice creates a *cylindrical shell* whose volume is approximately $2\pi r(x)h(x)\Delta x$.

The total volume is therefore approximated by the Riemann sum

$$\sum 2\pi r(x)h(x)\Delta x,$$

and we obtain the exact volume by letting $\Delta x \to 0$.

Thus the volume of S is

$$\boxed{\mathrm{Vol}(S) = 2\pi \int_a^b r(x)h(x)\,dx}.$$

Remark. Note that

$$r(x) = |x - A| = \begin{cases} x - A & \text{if } R \text{ is to the right of the axis,} \\ A - x & \text{if } R \text{ is to the left of the axis.} \end{cases}$$

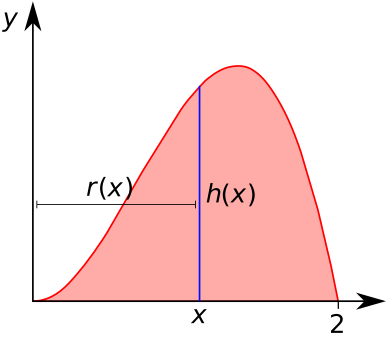
Examples

Let's now go back to our motivating example.

Example 1

Find the volume of the solid obtained by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y-axis.

Solution.



The height of each shell is

$$h(x) = y = 2x^2 - x^3,$$

and the radius of each shell is just

$$r(x) = x.$$

Therefore the volume is given by

$$\begin{aligned} 2\pi \int_0^2 x(2x^2 - x^3)\, dx &= 2\pi \int_0^2 2x^3 - x^4\, dx \\ &= 2\pi \left(\frac{x^4}{2} - \frac{x^5}{5} \Big|_0^2 \right) \\ &= 2\pi \left(8 - \frac{32}{5} \right) = \boxed{\frac{16\pi}{5}}. \end{aligned}$$

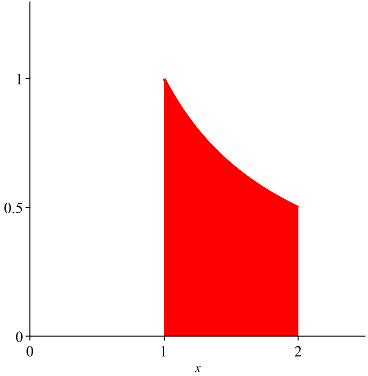
□

Remark. This was remarkably easier than trying to use cross sections!

Example 2

Find the volume of the solid obtained by rotating the region bounded by $y = 1/x$, $y = 0$, $x = 1$ and $x = 2$ about the y-axis.

Solution.



Cross sections are a bad idea since the formula for the outer radius depends on whether or not y is less than $1/2$.

However, the shell height is just $h(x) = y = 1/x$ and the shell radius is $r(x) = x$.

So the volume is given by

$$2\pi \int_1^2 x \cdot \frac{1}{x}\, dx = 2\pi \int_1^2 \, dx = \boxed{2\pi}.$$

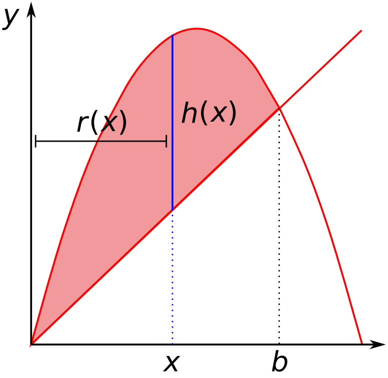
□

Remark. Although this problem is tractable using cross sections (unlike the previous one), observe how simple the shell approach was!

Example 3

Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = 4x - x^2$ about the y-axis.

Solution.



The shell height is

$$h(x) = (4x - x^2) - x$$
$$= 3x - x^2,$$

and the shell radius is

$$r(x) = x.$$

To find the value of b we equate the two functions:

$$x = 4x - x^2 \iff 3x = x^2 \iff x = 3.$$

Therefore the volume is

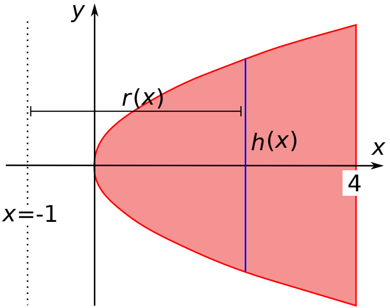
$$\begin{aligned}2\pi \int_0^3 x(3x - x^2)\,dx &= 2\pi \int_0^3 3x^2 - x^3\,dx \\ &= 2\pi \left(x^3 - \frac{x^4}{4} \Big|_0^3 \right) \\ &= 2\pi \left(27 - \frac{81}{4} \right) = \boxed{\frac{54\pi}{4}}.\end{aligned}$$

□

Example 4

Find the volume of the solid obtained by rotating the region enclosed by $x = y^2$ and $x = 4$ about the line $x = -1$.

Solution.



The shell height is

$$h(x) = \sqrt{x} - (-\sqrt{x}) \\ = 2\sqrt{x},$$

and the shell radius is

$$r(x) = x - (-1) = x + 1.$$

So the volume is

$$2\pi \int_0^4 2(x+1)\sqrt{x}\,dx = 4\pi \int_0^4 x^{3/2} + x^{1/2}\,dx$$

$$= 4\pi \left(\frac{x^{5/2}}{5/2} + \frac{x^{3/2}}{3/2} \Big|_0^4 \right)$$

$$= 4\pi \left(\frac{2x^{5/2}}{5} + \frac{2x^{3/2}}{3} \Big|_0^4 \right)$$

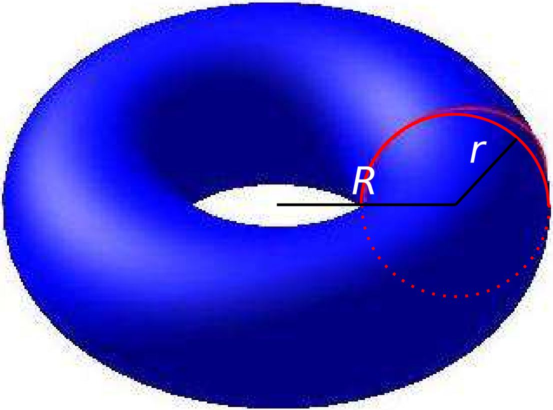
$$= 4\pi \left(\frac{2 \cdot 2^5}{5} + \frac{2 \cdot 2^3}{3} \right)$$

$$= 64\pi \left(\frac{4}{5} + \frac{1}{3} \right) = \boxed{\frac{1088\pi}{15}}.$$

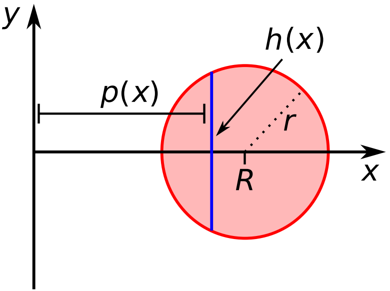
□

Example 5

Find the volume of a *torus* with radii r and R (see below).



We can obtain the torus by rotating a circle of radius r centered at $(R, 0)$ about the y-axis.



The equation of the circle is

$$(x - R)^2 + y^2 = r^2 \;\; \Rightarrow \;\; y = \pm\sqrt{r^2 - (x - R)^2}.$$

Therefore the shell height is

$$h(x) = 2\sqrt{r^2 - (x - R)^2},$$

while the shell radius is

$$p(x) = x.$$

The left and right endpoints of the circle occur where

$$x = R \pm r.$$

Thus

$$\text{Vol.} = 2\pi \int_{R-r}^{R+r} 2x\sqrt{r^2 - (x - R)^2}\, dx.$$

We start by making the substitution

$$u = x - R \;\; \Rightarrow \;\; du = dx, \;\; x = u + R,$$

which transforms the integral into

$$4\pi \int_{-r}^{r} (u + R)\sqrt{r^2 - u^2}\, du =$$

$$4\pi \left(\int_{-r}^{r} u\sqrt{r^2 - u^2}\, du + R \int_{-r}^{r} \sqrt{r^2 - u^2}\, du \right).$$

Because $u\sqrt{r^2 - u^2}$ is an odd function of u, and we are integrating it from $-r$ to r, the first integral vanishes.

The second integral represent the area under the graph of $y = \sqrt{r^2 - u^2}$ from $-r$ to r, which is just a semicircle. Hence the integral equals $\pi r^2/2$.

We conclude that

$$\text{Vol.} = 4\pi R \cdot \frac{\pi r^2}{2} = \boxed{2\pi^2 R r^2}.$$

$\square$

Remark. It is interesting to note that the volume of the torus can be written as

$$(2\pi R) \times (\pi r^2),$$

which is the circumference of the central interior circle multiplied by the area of the "tubular" cross section.