Integration by Parts

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Calculus II

Introduction

The substitution rule for integration is just the chain rule for derivatives in reverse.

If we “invert” the product rule instead, we get a useful integration technique known as *integration by parts*.

The utility of both substitution and integration by parts lies in the fact that they can be applied to *any* integral.

However, it takes a bit of experience to know *how* to apply them.

The Product Rule

The product rule for derivatives states that

$$\frac{d}{dx} f(x)g(x) = f(x)g'(x) + f'(x)g(x).$$

If we integrate both sides of this we obtain

$$f(x)g(x) = \int f(x)g'(x)\,dx + \int f'(x)g(x)\,dx.$$

Rearranging now gives

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}.$$

Integration by Parts

If we set

$$u = f(x) \qquad v = g(x)$$

$$du = f'(x)\,dx \qquad dv = g'(x)\,dx$$

our formula becomes

$$\boxed{\int u\,dv = uv - \int v\,du}.$$

This rule allows us to replace the integral on the left with the one on the right, which ideally is easier to evaluate.

Using this formula to compute an integral is called *integration by parts* or *partial integration*.

Remarks

- In principle, integration by parts can be applied to any integral simply by factoring the integrand in some way.
- Given $\int f(x)g(x)\,dx$, we set $u = f(x)$, $dv = g(x)\,dx$, compute du (this requires differentiation) and v (this requires integration), and apply the formula.
- Since we can typically factor a given function in a number of ways, this makes integration by parts very flexible.
- The skill lies in knowing which factorization is going to be fruitful.
- Although there are exceptions, one typically chooses u so that it gets simpler with differentiation and dv so that it doesn't get worse with integration.

Examples

Example 1

Compute $\int xe^{-x}\,dx$.

Solution. We choose the parts

$$u = x \qquad dv = e^{-x}\,dx$$

$$du = dx \qquad v = -e^{-x}$$

so that

$$\int xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx.$$

The new integral is easy to compute, and we obtain

$$\boxed{-xe^{-x} - e^{-x} + C}.$$

□

Example 2

Compute $\int x^2 e^{-x/3}\, dx$.

Solution. We choose the parts

$$u = x^2 \qquad dv = e^{-x/3}\, dx$$

$$du = 2x\, dx \qquad v = -3e^{-x/3}.$$

Integration by parts yields

$$\int x^2 e^{-x/3}\,dx = -3x^2 e^{-x/3} + 6\int x e^{-x/3}\,dx.$$

We've reduced the degree of x in the integral, but we're not finished.

We integrate by parts again, now taking

$$u = x \qquad dv = e^{-x/3}\,dx$$

$$du = dx \qquad v = -3e^{-x/3}.$$

The RHS above the becomes

$$-3x^2 e^{-x/3} + 6\left(-3xe^{-x/3} + 3\int e^{-x/3}\,dx\right).$$

$$= -3x^2e^{-x/3} - 18xe^{-x/3} - 54e^{-x/3} + C$$
$$= \boxed{(-3x^2 - 18x - 54)e^{-x/3} + C}.$$

□

In general, for an integral of the form

$$\int x^k e^{Ax}\,dx, \quad k \in \mathbb{N}, \quad A \in \mathbb{R},$$

choose $u = x^k$, $dv = e^{Ax}\,dx$ and integrate by parts repeatedly.

Remarks.

- This works if we replace e^x by $\cos x$ or $\sin x$.
- We can also replace x^k with any polynomial in x.

Example 3

Compute $\int \frac{\ln x}{x^2}\,dx$.

Solution. We choose the parts

$$\begin{aligned} u &= \ln x & dv &= x^{-2}\,dx \\ du &= \tfrac{1}{x}\,dx & v &= -x^{-1}. \end{aligned}$$

We then have

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int x^{-2}\,dx = \boxed{-\frac{\ln x}{x} - \frac{1}{x} + C}.$$

□

Example 4

Compute $\displaystyle\int \ln x\, dx$.

Solution. We choose the parts

$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x}\, dx \qquad v = x.$$

Integration by parts gives

$$\int \ln x\, dx = x \ln x - \int dx = \boxed{x \ln x - x + C}.$$

$\square$

Example 5

Compute $\int x(\ln x)^2\,dx$.

Solution. We choose the parts

$$u = (\ln x)^2 \qquad dv = x\,dx$$

$$du = \frac{2\ln x}{x}\,dx \qquad v = \frac{x^2}{2}.$$

We then have

$$\int x(\ln x)^2\,dx = \frac{x^2(\ln x)^2}{2} - \int x\ln x\,dx.$$

We integrate by parts again with

$$u = \ln x \qquad dv = x\,dx$$

$$du = \tfrac{1}{x}\,dx \qquad v = \tfrac{x^2}{2}.$$

This gives us

$$\frac{x^2(\ln x)^2}{2} - \left(\frac{x^2 \ln x}{2} - \frac{1}{2}\int x\,dx\right) = \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} + C$$
$$= \boxed{x^2\left(\frac{(\ln x)^2}{2} - \frac{\ln x}{2} + \frac{1}{4}\right) + C}.$$

$\square$

Remarks

For an integral of the form

$$\int x^A (\ln x)^k \, dx, \;\; k \in \mathbb{N}, \;\; A \in \mathbb{R},$$

use integration by parts with $u = (\ln x)^k$, $dv = x^A \, dx$, and repeat.

- This also works if we replace $\ln x$ by $\arcsin x$ or $\arccos x$, or $\ln x$ by a polynomial in $\ln x$.
- If we substitute $t = \ln x$, then $dt = \frac{dx}{x}$ and $x = e^t$. Thus

$$\int x^A (\ln x)^k \, dx = \int x^{A+1} (\ln x)^k \frac{dx}{x} = \int t^k e^{(A+1)t} \, dt,$$

which can be handled as above.

Example 6

Compute $\int e^x \cos x \, dx$.

Solution. We choose the parts

$$u = \cos x \qquad dv = e^x \, dx$$

$$du = -\sin x \, dx \qquad v = e^x$$

so that

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx.$$

Now take

$$u = \sin x \qquad dv = e^x\,dx$$

$$du = \cos x\,dx \qquad v = e^x.$$

The above then becomes

$$\int e^x \cos x\,dx = e^x \cos x + e^x \sin x - \int e^x \cos x\,dx.$$

We've arrived back at the integral we started with! Solving for it algebraically we obtain

$$\int e^x \cos x\,dx = \boxed{\frac{1}{2}e^x(\cos x + \sin x) + C}.$$

□

Example 7

Compute $\displaystyle\int \frac{x}{\sqrt{1+2x}}\,dx$.

Solution. We choose the parts

$$u = x \qquad dv = (1+2x)^{-1/2}dx$$

$$du = dx \qquad v = (1+2x)^{1/2},$$

which gives us

$$\int \frac{x}{\sqrt{1+2x}}\,dx = x\sqrt{1+2x} - \int (1+2x)^{1/2}\,dx$$

$$= \boxed{x\sqrt{1+2x} - \frac{1}{3}(1+2x)^{3/2} + C}.$$

Example 8

Compute $\displaystyle\int x^3\sqrt{1+x^2}\,dx$.

Solution. We choose the parts

$$u = x^2 \qquad dv = x(1+x^2)^{1/2}dx$$

$$du = 2x\,dx \qquad v = \tfrac{1}{3}(1+x^2)^{3/2},$$

so that

$$\int x^3\sqrt{1+x^2}\,dx = \frac{1}{3}x^2(1+x^2)^{3/2} - \frac{2}{3}\int x(1+x^2)^{3/2}\,dx$$

$$= \boxed{\frac{1}{3}x^2(1+x^2)^{3/2} - \frac{2}{15}(1+x^2)^{5/2} + C}.$$

Example 9

Compute $\int e^{\sqrt{x}}\,dx$.

Solution. We perform the preliminary substitution

$$t = \sqrt{x}, \;\; dt = \frac{1}{2\sqrt{x}}\,dx \quad \Rightarrow \quad dx = 2\sqrt{x}\,dt = 2t\,dt.$$

This gives us

$$\int e^{\sqrt{x}}\,dx = 2\int te^{t}\,dt,$$

which we can integrate by parts.

We choose the parts

$$u = t \qquad dv = e^t\,dt$$

$$du = dt \qquad v = e^t,$$

so that

$$\begin{aligned} 2\int te^t\,dt &= 2\left(te^t - \int e^t\,dt\right) \\ &= 2te^t - 2e^t + C \\ &= \boxed{2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C}. \end{aligned}$$

$\square$

Remarks

- The integrals $\int \frac{x}{\sqrt{1+2x}}\,dx$ and $\int x^3\sqrt{1+x^2}\,dx$ can also be computed using substitutions. It is interesting to compare the results of both computations.

- We can avoid the substitution in the preceding example by writing
$$\int e^{\sqrt{x}}\,dx = \int \sqrt{x}\frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx,$$
and then choosing $u = \sqrt{x}$, $dv = e^{\sqrt{x}}/\sqrt{x}\,dx$.

Partial Integration of Definite Integrals

Example 10

Evaluate $\displaystyle\int_0^\pi t \sin 3t \, dt$.

Solution. We choose the parts

$$u = t \qquad dv = \sin 3t \, dt$$

$$du = dt \qquad v = -\tfrac{1}{3}\cos 3t \, dt,$$

which gives us

$$\int_0^\pi t \sin 3t \, dt = -\frac{1}{3} t \cos 3t \Big|_0^\pi + \frac{1}{3}\int_0^\pi \cos 3t \, dt = \frac{\pi}{3} + \frac{1}{9}\sin 3t \Big|_0^\pi = \boxed{\pi/3}.$$

□

Tabular Integration by Parts

It is possible to implement repeated integration by parts using a table in order to speed up computations.

We will illustrate the tabular technique with a few examples.

Example 11

Compute $\displaystyle\int x^4 e^x \, dx$.

Solution. We choose the usual parts:

$$u = x^4, \;\; dv = e^x \, dx.$$

Build a table by repeatedly differentiating u and repeatedly integrating dv, stopping when the derivative on the left is zero.

$u = x^4$		$dv = e^x\,dx$
	$+$	
$4x^3$		e^x
	$-$	
$12x^2$		e^x
	$+$	
$24x$		e^x
	$-$	
24		e^x
	$+$	
0		e^x

Now multiply diagonally as shown, then alternately add and subtract the results:

$$x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x.$$

Add the constant of integration, and we're finished:

$$\boxed{e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) + C}$$

$\square$

Example 12

Compute $\displaystyle\int x^3 \sin 2x\, dx$.

Solution. We start by choosing the appropriate parts:

$$u = x^3, \quad dv = \sin 2x\, dx.$$

Then we construct our table.

$u = x^3$		$dv = \sin 2x\, dx$
	$+$	
$3x^2$		$-\frac{1}{2}\cos 2x$
	$-$	
$6x$		$-\frac{1}{4}\sin 2x$
	$+$	
6		$\frac{1}{8}\cos 2x$
	$-$	
0		$\frac{1}{16}\sin 2x$

Cross multiplication and addition/subtraction give us

$$-\frac{1}{2}x^3\cos 2x + \frac{3}{4}x^2\sin 2x + \frac{3}{4}x\cos 2x - \frac{3}{8}\sin 2x + C$$

$$= \boxed{\left(-\frac{x^3}{2} + \frac{3x}{4}\right)\cos 2x + \left(\frac{3x^2}{4} - \frac{3}{8}\right)\sin 2x + C}.$$

Remarks

- You can terminate the tabular process at *any* row provided you also include an appropriate integral term in your sum. If the final row of your table is
$$f_n(x) \qquad g_n(x),$$
you must include $\pm \int f_n(x) g_n(x)\,dx$, where the sign follows the alternation rule above.
- The tabular method can't handle every repeated integration by parts. For instance, we know the integral
$$\int x^2 (\ln x)^3\,dx$$
requires repeated integration by parts starting with $u = (\ln x)^3$, $dv = x^2\,dx$, but the derivatives of u get progressively *worse*.