Trigonometric Integrals

Ryan C. Daileda

Trinity University

Calculus II

Introduction

We will now begin studying techniques for integrating specific classes of functions.

Today we will be interested in integrands that are polynomial/rational expressions in sine and cosine.

Although it is possible to "rationalize" any such integrand, in many cases trig. identities and simple substitutions suffice.

We will motivate each of our general strategies with concrete examples.

Integrals of the Form $\int \cos^m x \sin^n x \, dx$

Example 1

Compute $\displaystyle\int \cos^2 x \sin^3 x \, dx$.

Solution. We "peel off" a factor of $\sin x$ and express everything else in terms of $\cos x$, using the fundamental identity

$$\cos^2 x + \sin^2 x = 1 \;\Rightarrow\; \sin^2 x = 1 - \cos^2 x.$$

This yields

$$\begin{aligned} \int \cos^2 x \sin^3 x \, dx &= \int \cos^2 x \sin^2 x \sin x \, dx \\ &= \int \cos^2 x (1 - \cos^2 x) \sin x \, dx. \end{aligned}$$

Now substitute $u = \cos x$, $du = -\sin x\, dx$:

$$\int \cos^2 x(1 - \cos^2 x) \sin x\, dx = -\int u^2(1 - u^2)\, du$$

$$= \int u^4 - u^2\, du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \boxed{\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C}$$

□

Example 2

Compute $\displaystyle\int \cos^5 x \sin^2 x \, dx$.

Solution. This time we "peel off" $\cos x$ and write what remains using only $\sin x$:

$$\begin{aligned}\int \cos^5 x \sin^2 x \, dx &= \int \cos^4 x \sin^2 x \cos x \, dx \\ &= \int (\cos^2 x)^2 \sin^2 x \cos x \, dx \\ &= \int (1 - \sin^2 x)^2 \sin^2 x \cos x \, dx.\end{aligned}$$

Now substitute $u = \sin x$, $du = \cos x \, dx$.

$$\begin{aligned}
\int (1-\sin^2 x)^2 \sin^2 x \cos x \, dx &= \int (1-u^2)^2 u^2 \, du \\
&= \int (1-2u^2+u^4) u^2 \, du \\
&= \int u^2 - 2u^4 + u^6 \, du \\
&= \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} + C \\
&= \boxed{\frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C}.
\end{aligned}$$

□

Example 3

Compute $\int \sin^5 x\,dx$.

Solution. We "peel off" $\sin x$, convert to $\cos x$, and substitute $u = \cos x$, $du = -\sin x\,dx$:

$$\int \sin^5 x\,dx = \int (1-\cos^2 x)^2 \sin x\,dx = -\int (1-u^2)^2\,du$$

$$= \int -1 + 2u^2 - u^4\,du = -u + \frac{2u^3}{3} - \frac{u^5}{5} + C$$

$$= \boxed{-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C}$$

□

General Strategy

Given an integral of the form $\int \cos^m x \sin^n x \, dx$:

- If m is odd, factor out $\cos x$, use $\cos^2 x = 1 - \sin^2 x$ in what remains, and substitute $u = \sin x$.

- If n is odd, factor out $\sin x$, use $\sin^2 x = 1 - \cos^2 x$ in what remains, and substitute $u = \cos x$.

Remark. We usually assume that m and n are both integers, but this method works as long as *at least one* of m and n is an odd number (the other can be any real number).

Even Powers

If both m and n are even, we instead make use of the half-angle formulas:

$$\cos^2 x = \frac{1+\cos 2x}{2} \quad \text{and} \quad \sin^2 x = \frac{1-\cos 2x}{2}.$$

Example 4

Compute $\displaystyle\int \cos^2 x \sin^2 x \, dx$.

Solution. We apply the half-angle formulas:

$$\int \cos^2 x \sin^2 x \, dx = \int \left(\frac{1+\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right) dx.$$

Expanding everything gives

$$\int \left(\frac{1+\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right)\,dx = \frac{1}{4}\int 1-\cos^2 2x\,dx.$$

Now we apply the half-angle formula again:

$$\begin{aligned}\frac{1}{4}\int 1-\cos^2 2x\,dx &= \frac{1}{4}\int 1-\frac{1+\cos 4x}{2}\,dx \\ &= \frac{1}{4}\int \frac{1}{2}-\frac{\cos 4x}{2}\,dx \\ &= \frac{1}{8}\left(x-\frac{1}{4}\sin 4x\right)+C \\ &= \boxed{\frac{x}{8}-\frac{\sin 4x}{32}+C}\end{aligned}$$

□

Example 5

Compute $\int \cos^4 x\, dx$.

Solution. Using the half-angle formula we have

$$\begin{aligned}
\int \cos^4 x\, dx &= \int (\cos^2 x)^2\, dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \\
&= \frac{1}{4}\int 1 + 2\cos 2x + \cos^2 2x\, dx \\
&= \frac{1}{4}\int 1 + 2\cos 2x + \frac{1+\cos 4x}{2}\, dx \\
&= \frac{1}{4}\int \frac{3}{2} + 2\cos 2x + \frac{\cos 4x}{2}\, dx
\end{aligned}$$

$$= \frac{1}{4}\left(\frac{3x}{2} + \sin 2x + \frac{\sin 4x}{8}\right) + C = \boxed{\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C}$$

$\square$

In General: Given $\int \cos^m x \sin^n x\, dx$ with *both* m and n even, write

$$\begin{aligned}\cos^m x \sin^n x &= (\cos^2 x)^{m/2} (\sin^2 x)^{n/2} \\ &= \left(\frac{1+\cos 2x}{2}\right)^{m/2} \left(\frac{1-\cos 2x}{2}\right)^{n/2},\end{aligned}$$

expand, and apply the procedures above.

Integrals of the Form $\int \tan^m x \sec^n x \, dx$

Here we need the fundamental trig. identity in the form

$$\cos^2 x + \sin^2 x = 1 \quad \Leftrightarrow \quad 1 + \tan^2 x = \sec^2 x,$$

and we recall the differentiation formulas

$$\frac{d}{dx} \tan x = \sec^2 x \quad \text{and} \quad \frac{d}{dx} \sec x = \sec x \tan x.$$

Example 6

Compute $\displaystyle\int \tan^4 x \sec^6 x \, dx$.

Solution. This time we factor out $\sec^2 x$ and express what remains in terms of $\tan x$:

$$\int \tan^4 x \sec^6 x \, dx = \int \tan^4 x (\sec^2 x)^2 \sec^2 x \, dx$$

$$= \int \tan^4 x(1 + \tan^2 x)^2 \sec^2 x \, dx.$$

Now substitute $u = \tan x$, $du = \sec^2 x \, dx$:

$$= \int u^4(1 + u^2)^2 \, du = \int u^4(1 + 2u^2 + u^4) \, du$$

$$= \int u^4 + 2u^6 + u^8 \, du = \frac{u^5}{5} + \frac{2u^7}{7} + \frac{u^9}{9} + C$$

$$= \boxed{\frac{1}{5}\tan^5 x + \frac{2}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C}$$

□

Example 7

Compute $\int \tan^3 x \sec x \, dx$.

Solution. Here we factor out $\tan x$, pair it with $\sec x$, end express everything else in terms of $\sec x$:

$$\int \tan^3 x \sec x \, dx = \int \tan^2 x \sec x \tan x \, dx$$
$$= \int (\sec^2 x - 1) \sec x \tan x \, dx.$$

Now substitute $u = \sec x$, $du = \sec x \tan x \, dx$:

$$= \int u^2 - 1 \, du = \frac{u^3}{3} - u + C = \boxed{\frac{1}{3}\sec^3 x - \sec x + C}$$

□

General Strategy

Given an integral of the form $\int \tan^m x \sec^n x \, dx$:

- If n is even, factor out $\sec^2 x$, use $\sec^2 x = 1 + \tan^2 x$ and substitute $u = \tan x$.
- If m is odd, factor out $\tan x$ and pair it with $\sec x$, use $\tan^2 x = 1 - \sec^2 x$ and substitute $u = \sec x$.

Remarks.

- Other situations require case-by-case use of trig. identities, substitution and integration by parts.
- One can also use the *rationalizing substitution* $t = \tan x/2$ to convert any trig. integral into the integral of a rational function.

Example 8

Compute $\int \tan^2 x \, dx$.

Solution. We just use the fundamental relationship:

$$\int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx = \boxed{\tan x - x + C}.$$

□

Example 9

Compute $\int \sec x \, dx$.

Solution. We notice that

$$\left.\begin{matrix} \tan x \\ \\ \sec x \end{matrix}\right\} \xrightarrow{\frac{d}{dx}} \begin{cases} \sec x \sec x \\ \\ \sec x \tan x. \end{cases}$$

If we add these we see that

$$\frac{d}{dx}(\sec x + \tan x) = \sec x \sec x + \sec x \tan x = \sec x(\sec x + \tan x).$$

Thus

$$\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \, dx = \int \frac{\frac{d}{dx}(\sec x + \tan x)}{\sec x + \tan x} \, dx.$$

If we make the substitution $u = \sec x + \tan x$ this becomes

$$\int \frac{du}{u} = \ln |u| + C = \boxed{\ln |\sec x + \tan x| + C}.$$

$\square$

Remark. This result is worth memorizing.

Example 10

Compute $\int \sec^3 x\, dx$.

Solution. We integrate by parts, choosing

$$u = \sec x \qquad dv = \sec^2 x\, dx$$

$$du = \sec x \tan x\, dx \qquad v = \tan x.$$

This gives us

$$\begin{aligned}\int \sec^3 x\, dx &= \sec x \tan x - \int \tan^2 x \sec x\, dx \\ &= \sec x \tan x - \int (\sec^2 x - 1) \sec x\, dx.\end{aligned}$$

Thus

$$\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx.$$

Solving algebraically for the integral and using the previous example we obtain

$$\int \sec^3 x\,dx = \boxed{\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C}.$$

□

Remark. We have analogous procedures/results if the pair $\{\sec x, \tan x\}$ is replaced by $\{\csc x, \cot x\}$.

Example 11

Compute $\int x \sec x \tan x \, dx$.

Solution. We integrate by parts, this time with

$$u = x \qquad dv = \sec x \tan x \, dx$$

$$du = dx \qquad v = \sec x.$$

Hence

$$\int x \sec x \tan x \, dx = x \sec x - \int \sec x \, dx$$

$$= \boxed{x \sec x - \ln|\sec x + \tan x| + C}.$$

□

Products of Functions with Different Periods

In this situation we use the identities

$$\sin A \cos B = \frac{1}{2}(\sin(A - B) + \sin(A + B)),$$
$$\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B)),$$
$$\cos A \cos B = \frac{1}{2}(\cos(A - B) + \cos(A + B)).$$

Example 12

Compute $\int \sin 3x \cos 5x \, dx$.

Solution. Using the first identity above gives

$$\int \sin 3x \cos 5x \, dx = \frac{1}{2} \int \sin(-2x) + \sin 8x \, dx$$

$$= \frac{1}{2}\int -\sin 2x + \sin 8x\, dx = \boxed{\frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C}.$$

$\square$

Remark. Another approach is to introduce complex numbers and use the formulas

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i},$$

together with the substitution $z = e^{ix}$, $dz = ie^{ix}dx$, $dx = dz/iz$.