Trigonometric Substitution

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Calculus II

Introduction

Using special substitutions, certain integrals involving quadratic polynomials can be transformed into trigonometric integrals.

These substitutions are somewhat different than what we are used to. Given $\int f(x)\,dx$, we typically substitute $u = g(x)$. Here we will make the substitution $x = g(\theta)$.

We start by considering integrands involving quadratic polynomials of the form $Ax^2 + B$. General quadratic polynomials can be handled by completing the square.

Motivating Example

Example 1

Compute $\displaystyle\int \frac{dx}{x^2\sqrt{25-x^2}}$.

Solution. We make the substitution

$$x = 5\sin\theta, \;\; dx = 5\cos\theta\, d\theta.$$

This transforms the integral into

$$\int \frac{dx}{x^2\sqrt{25-x^2}} = \int \frac{5\cos\theta\, d\theta}{25\sin^2\theta\sqrt{25-25\sin^2\theta}}$$

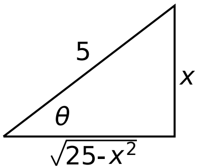
$$= \frac{1}{25}\int \frac{\cos\theta\, d\theta}{\sin^2\theta\sqrt{1-\sin^2\theta}}$$

$$= \frac{1}{25}\int \frac{\cos\theta\, d\theta}{\sin^2\theta\cos\theta}$$

$$= \frac{1}{25}\int \csc^2\theta\, d\theta = -\frac{1}{25}\cot\theta + C.$$

To back substitute we draw a right triangle:

$$x = 5\sin\theta \;\Rightarrow\; \sin\theta = \tfrac{x}{5} \;\Rightarrow$$



We use the Pythagorean theorem to find the length of the third side.

This tells us that

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{25-x^2}/5}{x/5} = \frac{\sqrt{25-x^2}}{x}.$$

Thus

$$\int \frac{dx}{x^2\sqrt{25-x^2}} = -\frac{1}{25}\cot\theta + C = \boxed{-\frac{\sqrt{25-x^2}}{25x} + C}.$$

□

Idea. Substitute a trig. function for x to get rid of radicals and convert to a trig. integral.

General Strategy

In general, we have the following guidelines for trigonometric substitution.

Integrand Involves	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$\cos^2\theta = 1 - \sin^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\tan^2\theta = \sec^2 - 1\theta$
$\sqrt{x^2 + a^2}$	$x = a\tan\theta$	$\sec^2\theta = \tan^2\theta + 1$

In each case, after the substitution the radical will become a trigonometric function, e.g. if we set $x = a\sec\theta$, then

$$\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = a\sqrt{\sec^2\theta - 1} = a\tan\theta.$$

Remarks

- When making a trigonometric substitution, be sure to compute dx appropriately. For example:

  $$x = a\tan\theta \quad \Rightarrow \quad dx = a\sec^2\theta\, d\theta.$$

  In other words, $dx \neq d\theta$. It is *very easy* to forget this!

- Back substitution almost always involves drawing a right triangle that expresses the geometric relationship between θ and x.

- Trigonometric substitutions can be useful any time the quantities $a^2 - x^2$, $x^2 - a^2$ and $x^2 + a^2$ appear in an integrand, regardless of what power they are raised to, e.g. $(1 - x^2)^{5/2}$.

Examples

Example 2

Compute $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-9}}$.

Solution. We substitute

$$x = 3\sec\theta, \quad dx = 3\sec\theta\tan\theta\, d\theta.$$

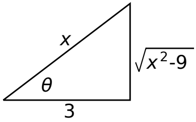
This yields

$$\int \frac{dx}{x^2\sqrt{x^2-9}} = \int \frac{3\sec\theta\tan\theta\, d\theta}{9\sec^2\theta\sqrt{9\sec^2\theta-9}} = \frac{1}{9}\int \frac{\tan\theta\, d\theta}{\sec\theta\tan\theta}$$

$$= \frac{1}{9} \int \cos\theta \, d\theta = \frac{\sin\theta}{9} + C.$$

We now draw a triangle:

$$x = 3\sec\theta \;\; \Rightarrow \;\; \cos\theta = \frac{3}{x} \;\; \Rightarrow$$



This tells us that

$$\frac{\sin\theta}{9} + C = \boxed{\frac{\sqrt{x^2 - 9}}{9x} + C}.$$

Example 3

Compute $\displaystyle\int \frac{dx}{x\sqrt{4+x^2}}$.

Solution. Now we substitute

$$x = 2\tan\theta, \quad dx = 2\sec^2\theta\, d\theta.$$

The integral then becomes

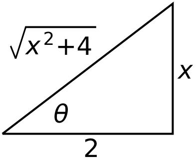
$$\int \frac{dx}{x\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{2\tan\theta\sqrt{4+4\tan^2\theta}} = \frac{1}{2}\int \frac{\sec^2\theta\, d\theta}{\tan\theta\sec\theta}$$

$$= \frac{1}{2}\int \frac{\sec\theta}{\tan\theta}\,d\theta = \frac{1}{2}\int \frac{1}{\sin\theta}\,d\theta = \frac{1}{2}\int \csc\theta\,d\theta$$

$$= \frac{1}{2}\ln|\csc\theta - \cot\theta| + C.$$

Now we draw our triangle:

$$x = 2\tan\theta \ \Rightarrow \ \tan\theta = \tfrac{x}{2} \ \Rightarrow$$



Thus

$$\frac{1}{2}\ln|\csc\theta - \cot\theta| + C = \boxed{\frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}-2}{x}\right| + C}.$$

Example 4

Evaluate $\displaystyle\int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}}\,dx.$

Solution. We set $5x = 3\sin\theta$ or

$$x = \frac{3}{5}\sin\theta, \quad dx = \frac{3}{5}\cos\theta\,d\theta.$$

Thus

$$\int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}}\,dx = \int_0^{\pi/2} \frac{(9/25)\sin^2\theta\cdot(3/5)\cos\theta\,d\theta}{\sqrt{9-9\sin^2\theta}}$$

$$= \frac{9}{125}\int_0^{\pi/2} \frac{\sin^2\theta\cos\theta}{\cos\theta}\,d\theta$$

$$= \frac{9}{125}\int_0^{\pi/2} \sin^2\theta\, d\theta = \frac{9}{125}\int_0^{\pi/2} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \frac{9}{250}\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/2} = \frac{9}{250}\left(\frac{\pi}{2} - 0\right) = \boxed{\frac{9\pi}{500}}.$$

□

Remark. Note that because we transformed the limits of integration when we performed the substitution, using a triangle for back substitution isn't necessary.

We can handle an arbitrary quadratic polynomial under a radical sign by first completing the square.

Example 5

Compute $\int \frac{x^2}{\sqrt{x^2+4x-5}}\,dx$.

Solution. Completing the square gives us

$$\int \frac{x^2}{\sqrt{x^2+4x-5}}\,dx = \int \frac{x^2}{\sqrt{(x+2)^2-9}}\,dx.$$

So we make the preliminary substitution

$$t = x+2, \;\; dt = dx, \;\; x = t-2.$$

This yields

$$\int \frac{x^2}{\sqrt{(x+2)^2-9}}\,dx = \int \frac{(t-2)^2}{\sqrt{t^2-9}}\,dt = \int \frac{t^2-4t+4}{\sqrt{t^2-9}}\,dt$$

$$= \int \frac{-4t}{\sqrt{t^2-9}}\,dt + \int \frac{t^2+4}{\sqrt{t^2-9}}\,dt.$$

In the first integral we substitute $u = t^2 - 9$, $du = 2t\,dt$ and in the second we set $t = 3\sec\theta$, $dt = 3\sec\theta\tan\theta\,d\theta$. This gives us

$$\int \frac{-2}{\sqrt{u}}\,du + 3\int \frac{(9\sec^2\theta + 4)\sec\theta\tan\theta}{\sqrt{9\sec^2\theta - 9}}\,d\theta$$

$$= -4\sqrt{u} + \int 9\sec^3\theta + 4\sec\theta\,d\theta$$

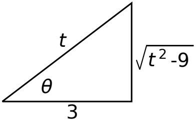
$$= -4\sqrt{t^2-9} + \frac{9}{2}\sec\theta\tan\theta + \frac{9}{2}\ln|\sec\theta + \tan\theta|$$

$$+ 4\ln|\sec\theta + \tan\theta| + C$$

$$= -4\sqrt{t^2 - 9} + \frac{9}{2}\sec\theta\tan\theta + \frac{17}{2}\ln|\sec\theta + \tan\theta| + C.$$

Now draw a triangle:

$$t = 3\sec\theta \;\Rightarrow\; \cos\theta = \tfrac{3}{t} \;\Rightarrow$$



Our result then becomes

$$-4\sqrt{t^2 - 9} + \frac{9}{2}\cdot\frac{t}{3}\cdot\frac{\sqrt{t^2 - 9}}{3} + \frac{17}{2}\ln\left|\frac{t}{3} + \frac{\sqrt{t^2 - 9}}{3}\right| + C$$

$$= \sqrt{t^2-9}\left(\frac{t}{2}-4\right)+\frac{17}{2}\ln\left|t+\sqrt{t^2-9}\right|+C,$$

where we have used laws of logarithms and absorbed $-\frac{17}{2}\ln 3$ into C.

Finally, we back substitute $t = x+2$:

$$\sqrt{(x+2)^2-9}\left(\frac{x+2}{2}-4\right)+\frac{17}{2}\ln\left|x+2+\sqrt{(x+2)^2-9}\right|+C$$

$$= \boxed{\sqrt{x^2+4x-5}\left(\frac{x}{2}-3\right)+\frac{17}{2}\ln\left|x+2+\sqrt{x^2+4x-5}\right|+C}.$$

$\square$

Remark. Notice that the quantity under the radicals in the antiderivative matches the quantity under the radical in the integrand.