Rational Functions and Partial Fractions
Part 1

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Calculus II

Introduction

A *rational function* has the form

$$f(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials.

For instance

$$\frac{2x+1}{3x^2-6x+2}, \quad \frac{x^2-x-1}{2x-7}, \quad \frac{15x^2-3x^3}{2-x^2}, \quad \frac{x^2}{x^2-1}, \quad \frac{x}{x^3+1}$$

are all rational functions.

Goal. Develop a technique for integrating *any* rational function.

Partial Fractions

Fact. Every polynomial with real coefficients can be factored (uniquely) into the product of polynomials of the form

$$\underbrace{(ax+b)}_{\text{linear}}{}^n \quad \text{or} \quad \underbrace{(ax^2+bx+c)}_{\text{quadratic}}{}^n \quad (\underbrace{b^2-4ac}_{\text{discriminant}} < 0).$$

By factoring the denominator of a rational function, we can always express a given rational function as a sum of simpler rational functions of the form

$$\frac{A}{(ax+b)^n} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^n} \quad (b^2-4ac<0).$$

This is called the *partial fraction* decomposition.

Examples

There are a number os steps and cases involved. Let's look at an example.

Example 1

Compute $\displaystyle\int \frac{x^3+1}{x^2-x-6}\,dx$.

Solution.

Step 1. Perform polynomial long division (if necessary) so that

$$\deg(\text{numer.}) < \deg(\text{denom.}).$$

We have

$$
\begin{array}{r|llll}
 & x & +1 & & \\
\hline
x^2 - x - 6 & x^3 & +1 & & \\
 & x^3 & -x^2 & -6x & \\
\cline{2-4}
 & & x^2 & +6x & +1 \\
 & & x^2 & -x & -6
\end{array}
$$

We have

$$
\begin{array}{r|lllll}
 & x & +1 & & \\
\hline
x^2 - x - 6 & x^3 & +1 & & \\
 & x^3 & -x^2 & -6x & \\
\cline{2-4}
 & & x^2 & +6x & +1 \\
 & & x^2 & -x & -6 \\
\cline{3-5}
 & & & 7x & +7
\end{array}
$$

So the *quotient* is $x + 1$ and the *remainder* is $7x + 7$:

$$\frac{x^3 + 1}{x^2 - x - 6} = x + 1 + \frac{7x + 7}{x^2 - x - 6}.$$

Step 2. Factor the denominator. In this case

$$x^2 - x - 6 = (x-3)(x+2).$$

Step 3. Compute the *partial fraction decomposition*. Set

$$\frac{7x+7}{x^2-x-6} = \frac{7x+7}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$
$$= \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}.$$

Now equate the numerators:

$$7x+7 = A(x+2) + B(x-3).$$

To find A and B we need a system of equations. There are two options:

1. Collect common powers of x on the RHS and equate coefficients.
2. Plug in "convenient" x values.

In this case it is convenient to plug in $x = -2, 3$ (the roots of the denominator).

$$\underline{x = -2:}\ 7(-2) + 7 = -5B \quad \Rightarrow \quad B = \frac{7}{5},$$

$$\underline{x = 3:}\ 7(3) + 7 = 5A \quad \Rightarrow \quad A = \frac{28}{5}.$$

This tells us that

$$\frac{7x + 7}{x^2 - x - 6} = \frac{28/5}{x - 3} + \frac{7/5}{x + 2}.$$

Step 4. Integrate. We now have

$$\int \frac{x^3+1}{x^2-x-6}\,dx = \int x+1+\frac{7x+7}{x^2-x-6}\,dx$$

$$= \int x+1+\frac{28/5}{x-3}+\frac{7/5}{x+2}\,dx$$

$$= \boxed{\frac{x^2}{2}+x+\frac{28}{5}\ln|x-3|+\frac{7}{5}\ln|x+2|+C}.$$

$\square$

Example 2

Compute $\displaystyle\int \frac{x^4-2}{x^3-x}\,dx$.

Solution.

Step 1. Long division (if necessary).

$$\begin{array}{r|ll} & x & \\ \hline x^3-x & x^4 & -2 \\ & x^4 & -x^2 \\ \hline & & x^2 \quad -2 \end{array}$$

Thus

$$\frac{x^4-2}{x^3-x} = x + \frac{x^2-2}{x^3-x}.$$

Step 2. Factor the denominator. We have

$$x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1).$$

Step 3. Compute the partial fraction decomposition. Set

$$\frac{x^2 - 2}{x^3 - x} = \frac{x^2 - 2}{x(x - 1)(x + 1)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{x + 1}$$

$$= \frac{A(x - 1)(x + 1) + Bx(x + 1) + Cx(x - 1)}{x(x - 1)(x + 1)}$$

and equate the numerators:

$$x^2 - 2 = A(x - 1)(x + 1) + Bx(x + 1) + Cx(x - 1).$$

Plugging in $x = 0, 1, -1$ we obtain

$$\begin{aligned} -2 = -A &\Rightarrow A = 2, \\ -1 = 2B &\Rightarrow B = \frac{-1}{2}, \\ -1 = 2C &\Rightarrow C = \frac{-1}{2}. \end{aligned}$$

Thus

$$\frac{x^2 - 2}{x^3 - x} = \frac{2}{x} + \frac{-1/2}{x-1} + \frac{-1/2}{x+1}.$$

Step 4. Integrate. We now see that

$$\int \frac{x^4 - 2}{x^3 - x}\, dx = \int x + \frac{2}{x} + \frac{-1/2}{x-1} + \frac{-1/2}{x+1}\, dx$$

$$= \boxed{\frac{x^2}{2} + 2\ln|x| - \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C}.$$

□

In General. If

$$R(x) = \frac{P(x)}{(a_1x - b_1)(a_2x - b_2)\cdots(a_nx - b_n)}$$

with

- $\deg P(x) < n$,
- all $r_i = b_i/a_i$ (the roots of the denom.) *distinct*,

then the partial fraction decomposition has the form

$$R(x) = \frac{A_1}{a_1x - b_1} + \frac{A_2}{a_2x - b_2} + \cdots + \frac{A_n}{a_nx - b_n}.$$

To find the A_i's we add the fractions on the RHS and compare numerators:

$$P(x) = \sum_{i=1}^{n} A_i (a_1 x - b_1)(a_2 x - b_2) \cdots \underbrace{(a_i x - b_i)}_{\text{omit}} \cdots (a_n x - b_n).$$

Notice that if we take $x = r_j = b_j / a_j$ then we get

$$P(r_j) = A_j (a_1 r_j - b_1)(a_2 r_j - b_2) \cdots \underbrace{(a_j r_j - b_j)}_{\text{omit}} \cdots (a_n r_j - b_n)$$

and we find that

$$\boxed{A_j = \frac{P(r_j)}{(a_1 r_j - b_1)(a_2 r_j - b_2) \cdots \underbrace{(a_j r_j - b_j)}_{\text{omit}} \cdots (a_n r_j - b_n)}.$$

That is, we can find A_j by plugging r_j into $R(x)$ and "forgetting" the factor of $a_j r_j - b_j = 0$ in the denominator.

Example 3

Find the partial fraction decomposition of

$$R(x) = \frac{x^3 - x + 3}{(2x-1)(x+1)(4x-3)(x-2)}.$$

Solution. Because the degree of the numerator is < 4 and the denominator has distinct linear factors, we know that

$$\frac{x^3 - x + 3}{(2x-1)(x+1)(4x-3)(x-2)} = \frac{A}{2x-1} + \frac{B}{x+1} + \frac{C}{4x-3} + \frac{D}{x-2}.$$

We find A, B, C, D by plugging $x = 1/2, -1, 3/4, 2$ (respectively) into $R(x)$ and "forgetting" the factor of zero:

$$A = \frac{(1/2)^3 - (1/2) + 3}{\underbrace{(2(1/2) - 1)}_{\text{forget}}(1/2 + 1)(4(1/2) - 3)(1/2 - 2)} = \frac{21/8}{9/4} = \frac{7}{6},$$

$$B = \frac{(-1)^3 - (-1) + 3}{(2(-1) - 1)\underbrace{(-1 + 1)}_{\text{forget}}(4(-1) - 3)(-1 - 2)} = \frac{3}{-63} = \frac{-1}{21},$$

$$C = \frac{(3/4)^3 - (3/4) + 3}{(2(3/4) - 1)(3/4 + 1)\underbrace{(4(3/4) - 3)}_{\text{forget}}(3/4 - 2)} = \frac{171/64}{-35/32} = \frac{-171}{70},$$

$$D = \frac{2^3 - 2 + 3}{(2 \cdot 2 - 1)(2 + 1)(4 \cdot 2 - 3)\underbrace{(2 - 2)}_{\text{forget}}} = \frac{9}{45} = \frac{1}{5}.$$

Thus the partial fraction decomposition is

$$\boxed{\frac{7/6}{2x-1} - \frac{1/21}{x+1} - \frac{171/70}{4x-3} + \frac{1/5}{x-2}}.$$

$\square$

Remark. Now we can easily see that

$$\int \frac{x^3 - x + 3}{(2x-1)(x+1)(4x-3)(x-2)}\,dx$$
$$= \frac{7}{12}\ln|2x-1| - \frac{1}{21}\ln|x+1| - \frac{171}{280}\ln|4x-3| + \frac{1}{5}\ln|x-2| + C.$$

Repeated Linear Factors

If any of the factors of the denominator of a rational function are *repeated* (occur with exponent > 1), we must proceed a little differently.

Example 4

Compute $\displaystyle\int \frac{1}{(x+5)^2(x+1)}\,dx$.

Solution. In this case we set

$$\begin{aligned}\frac{1}{(x+5)^2(x+1)} &= \frac{A}{x+5} + \frac{B}{(x+5)^2} + \frac{C}{x+1}\\ &= \frac{A(x+1)(x+5) + B(x+1) + C(x+5)^2}{(x+5)^2(x+1)}.\end{aligned}$$

Equating the numerators tells us that

$$A(x+1)(x+5)+B(x+1)+C(x+5)^2=1.$$

Choosing convenient x values we find

$$\underline{x=-1:}\ 16C=1 \ \Rightarrow\ C=1/16,$$
$$\underline{x=-5:}\ -4B=1 \ \Rightarrow\ B=-1/4,$$
$$\underline{x=0:}\ 5A+B+25C=1 \ \Rightarrow\ A=\frac{1}{5}(1-B-25C)=-1/16.$$

Thus

$$\int \frac{1}{(x+5)^2(x+1)}\,dx=\int \frac{-1/16}{x+5}+\frac{-1/4}{(x+5)^2}+\frac{1/16}{x+1}\,dx$$

$$= -\frac{1}{16}\ln|x+5| + \frac{1}{4(x+5)} + \frac{1}{16}\ln|x+1| + C$$

$$= \boxed{\frac{1}{4(x+5)} + \frac{1}{16}\ln\left|\frac{x+1}{x+5}\right| + C}.$$

□

In General. If the denominator of a rational function includes a factor of the form $(ax+b)^n$, then its partial fraction decomposition must include

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$

Examples

Write out the partial fraction decompositions of the following rational functions.

$$\frac{x+3}{x^2(x-1)(x+2)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+2} + \frac{E}{(x+2)^2} + \frac{F}{(x+2)^3}$$

$$\frac{x^2-7x+2}{(x-3)^2(x+5)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{x+5} + \frac{D}{(x+5)^2}$$

$$\frac{x^5}{(2x+1)^3(x+1)}$$

$$= Ax + B + \frac{C}{x+1} + \frac{D}{2x+1} + \frac{E}{(2x+1)^2} + \frac{F}{(2x+1)^3}$$

Note that the final example would require polynomial long division to find A and B.