Rational Functions and Partial Fractions
Part 2

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Calculus II

Recall

Our goal is to develop a method for integrating rational functions using their *partial fraction decompositions* (PFDs).

In general, these have the form

$$R(x) = \frac{P(x)}{Q(x)} = q(x) + T_1(x) + T_2(x) + \cdots + T_n(x),$$

where:

- $P(x)$ and $Q(x)$ are polynomials;
- $q(x)$ is the quotient when $P(x)$ is divided by $Q(x)$;
- each $T_i(x)$ corresponds to one of the irreducible (linear or quadratic) factors of $Q(x)$.

If the linear factor $ax+b$ divides $Q(x)$ with multiplicity n, then the corresponding term in the PFD is

$$T(x) = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$

Such terms are easy to integrate using the logarithm or the power rule.

If, however, $Q(x)$ is divisible by ax^2+bx+c ($b^2-4ac<0$) with multiplicity n, then we must instead include

$$T(x) = \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}.$$

Examples

Write out the general form of the PFDs of the following rational functions.

$$\frac{3x}{x^3+1} = \frac{3x}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$\frac{1}{x(x^2+4)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

$$\frac{x^3}{x^4-1} = \frac{x^3}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$$\frac{3x^2-2x+1}{x^2(x-3)^2(x^2-x+5)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3} + \frac{D}{(x-3)^2} + \frac{Ex+F}{x^2-x+5} + \frac{Gx+H}{(x^2-x+5)^2}$$

Computing General PFDs

A general theorem in abstract algebra guarantees that PFDs of rational functions exist. The difficult part can be actually computing them.

Example 1

Find the PFD of $\dfrac{3x}{x^3+1}$.

Solution. Since $x^3+1=(x+1)(x^2-x+1)$, the PFD has the form

$$\frac{3x}{x^3+1}=\frac{3x}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$$

$$=\frac{A(x^2-x+1)+(Bx+C)(x+1)}{(x+1)(x^2-x+1)}.$$

Equating the numerators yields

$$3x = A(x^2 - x + 1) + (Bx + C)(x + 1).$$

Thus:

$$\underline{x = -1} \;\Rightarrow\; -3 = 3A \;\Rightarrow\; A = -1,$$

$$\underline{x = 0} \;\Rightarrow\; 0 = A + C \;\Rightarrow\; C = 1,$$

$$\underline{x = 1} \;\Rightarrow\; 3 = A + 2(B + C) \;\Rightarrow\; B = 1.$$

So the PFD is

$$\boxed{\frac{3x}{x^3+1} = \frac{-1}{x+1} + \frac{x+1}{x^2-x+1}}.$$

□

Example 2

Find the PFD of $\dfrac{9(x-1)}{x^2(x^2+9)}$.

Solution. The general form of the PFD is

$$\frac{9(x-1)}{x^2(x^2+9)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+9}$$

$$= \frac{Ax(x^2+9) + B(x^2+9) + (Cx+D)x^2}{x^2(x^2+9)}$$

Equating the numerators gives us

$$9(x-1) = Ax(x^2+9) + B(x^2+9) + (Cx+D)x^2.$$

Therefore:

$$\underline{x=0} \;\Rightarrow\; -9 = 9B \;\Rightarrow\; B = -1,$$

$$\underline{x=1} \;\Rightarrow\; 0 = 10A + 10B + C + D,$$

$$\underline{x=-1} \;\Rightarrow\; -18 = -10A + 10B - C + D,$$

$$\underline{x=2} \;\Rightarrow\; 9 = 26A + 13B + 8C + 4D.$$

Adding the second and the third equations yields

$$-18 = 20B + 2D = -20 + 2D \;\Rightarrow\; D = 1.$$

Putting $B = -1$ and $D = 1$ into the second and third equations gives us

$$\left.\begin{aligned} 10A + C &= 9 \\ 26A + 8C &= 18 \end{aligned}\right\} \;\Rightarrow\; A = 1, \;\; C = -1.$$

We therefore have the PFD

$$\boxed{\frac{9(x-1)}{x^2(x^2+9)} = \frac{1}{x} - \frac{1}{x^2} + \frac{1-x}{x^2+9}}.$$

□

Remark. We can solve for C and D more rapidly by using the *complex* argument $x = 3i$:

$$\underline{x = 3i} \;\Rightarrow\; 9(3i-1) = -9(3Ci + D) \;\Rightarrow\; -1 + 3i = -D - 3Ci.$$

Because C and D are both *real*, this immediately tells us that $D = 1$ and $C = -1$.

The first two equations (for $x = 0, 1$) are then sufficient to determine A and B.

Integrating General PFDs

Let's look at some examples first.

Example 3

Compute $\int \frac{9(x-1)}{x^2(x^2+9)}\,dx$.

Solution. Using the PFD we computed above, we have

$$\int \frac{9(x-1)}{x^2(x^2+9)}\,dx = \int \frac{1}{x} - \frac{1}{x^2} + \frac{1-x}{x^2+9}\,dx$$

$$= \ln|x| + \frac{1}{x} + \underbrace{\int \frac{dx}{x^2+9}}_{\text{use arctan}} - \underbrace{\int \frac{x}{x^2+9}\,dx}_{\text{sub. } u=x^2+9}.$$

The substitution $x = at$ can be used to derive the general integration formula

$$\boxed{\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C}.$$

Applying this above we find that

$$\int \frac{9(x-1)}{x^2(x^2+9)}\, dx = \boxed{\ln|x| + \frac{1}{x} + \frac{1}{3} \arctan\left(\frac{x}{3}\right) - \frac{1}{2} \ln(x^2+9) + C}.$$

$\square$

Remark. We don't need absolute values in the second logarithm since $x^2 + 9$ is never negative.

Example 4

Compute $\displaystyle\int \frac{3x}{x^3+1}\,dx$.

Solution. According to our work above, we have

$$\begin{aligned}\int \frac{3x}{x^3+1}\,dx &= \int \frac{-1}{x+1} + \frac{x+1}{x^2-x+1}\,dx \\ &= -\ln|x+1| + \int \frac{x+1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\,dx \\ &= -\ln|x+1| + 4\int \frac{x+1}{(2x-1)^2+3}\,dx.\end{aligned}$$

In the final step we multiplied the numerator and denominator by 4 to get rid of the fractional coefficients.

Now substitute $t = 2x - 1$, $dt = 2dx$:

$$-\ln|x+1| + 4\int \frac{x+1}{(2x-1)^2+3}\,dx = -\ln|x+1| + \int \frac{t+3}{t^2+3}\,dt$$

$$= -\ln|x+1| + \underbrace{\int \frac{t}{t^2+3}\,dt}_{\text{sub. } u=t^2+3} + 3\underbrace{\int \frac{dt}{t^2+3}}_{\text{use arctan}}$$

$$= -\ln|x+1| + \frac{1}{2}\ln(t^2+3) + \frac{3}{\sqrt{3}}\arctan\left(\frac{t}{\sqrt{3}}\right) + C$$

$$= -\ln|x+1| + \frac{1}{2}\ln\left((2x-1)^2+3\right) + \sqrt{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

$$= \boxed{-\ln|x+1| + \frac{1}{2}\ln\left(x^2-x+1\right) + \sqrt{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C}.$$

□

Quadratic Factors in General

What about the general integration of terms in a PFD coming from irreducible quadratic factors of the denominator?
Consider $ax^2 + bx + c$ with $b^2 - 4ac < 0$. Factor out a and complete the square:

$$\begin{aligned} a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) &= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right] \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]. \end{aligned}$$

Because $\frac{4ac-b^2}{4a^2}$ is positive, we can set

$$\frac{4ac - b^2}{4a^2} = \alpha^2.$$

The substitution $t = x + \frac{b}{2a}$, $dt = dx$ therefore yields

$$\int \frac{Ax+B}{(ax^2+bx+c)^k}\,dx = \int \frac{A't+B'}{a^k(t^2+\alpha^2)^k}\,dt$$
$$= \frac{A'}{a^k}\int \frac{t}{(t^2+\alpha^2)^k}\,dt + \frac{B'}{a^k}\int \frac{dt}{(t^2+\alpha^2)^k}.$$

The first of these can be handled with the substitution $u = t^2 + \alpha^2$.

If $k > 1$, in the second we can make the trigonometric substitution $t = \alpha \tan\theta$, $dt = \alpha \sec^2\theta$:

$$\int \frac{dt}{(t^2+\alpha^2)^k}\,dt = \int \frac{\alpha\sec^2\theta}{\alpha^{2k}\sec^{2k}\theta}\,d\theta = \frac{1}{\alpha^{2k-1}}\int \cos^{2(k-1)}\theta\,d\theta.$$

This even power of cosine can now be integrated by repeated application of the half-angle formula

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}.$$

After integration, this will yield terms involving

$$\cos 2^m\theta \quad \text{and} \quad \sin 2^m\theta.$$

These can be expressed in terms of $\cos\theta$ and $\sin\theta$ (and hence t) by repeatedly using the *double-angle formulas*

$$\cos 2\theta = \cos^2\theta - \sin^2\theta,$$
$$\sin 2\theta = 2\cos\theta\sin\theta.$$

Examples

Example 5

Compute $\int \frac{dx}{(x^2+4)^3}$.

Solution. Setting $x = 2\tan\theta$ and $dx = 2\sec^2\theta\, d\theta$, we have

$$\int \frac{dx}{(x^2+4)^3} = \int \frac{2\sec^2\theta}{4^3\sec^6\theta}\, d\theta = \frac{1}{32}\int \cos^4\theta\, d\theta$$

$$= \frac{1}{32}\int \left(\frac{1+\cos 2\theta}{2}\right)^2\, d\theta$$

$$= \frac{1}{128}\int 1 + 2\cos 2\theta + \cos^2 2\theta\, d\theta$$

$$= \frac{1}{128}\left(\theta + \sin 2\theta + \frac{1}{2}\int 1 + \cos 4\theta \, d\theta\right)$$

$$= \frac{1}{128}\left(\frac{3\theta}{2} + \sin 2\theta + \frac{\sin 4\theta}{8}\right) + C$$

$$= \frac{1}{128}\left(\frac{3\theta}{2} + 2\sin\theta\cos\theta + \frac{\sin 2\theta \cos 2\theta}{4}\right) + C$$

$$= \frac{1}{128}\left(\frac{3\theta}{2} + 2\sin\theta\cos\theta + \frac{(2\sin\theta\cos\theta)(\cos^2\theta - \sin^2\theta)}{4}\right) + C.$$

If we use the relationship $\tan\theta = \frac{x}{2}$ to draw a right triangle, we find that

$$\sin\theta = \frac{x}{\sqrt{x^2+4}} \quad \text{and} \quad \cos\theta = \frac{2}{\sqrt{x^2+4}}.$$

Substituting these into our result and simplifying we obtain

$$\int \frac{dx}{(x^2+4)^3} = \boxed{\frac{x}{16(x^2+4)^2} + \frac{3x}{128(x^2+4)} + \frac{3}{256}\arctan\left(\frac{x}{2}\right) + C}.$$

□

Remark. Another approach is to repeatedly use the *reduction formula*

$$\int \frac{dx}{(x^2+a^2)^n} = \frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}} + \frac{1}{a^2}\left(1 - \frac{1}{2(n-1)}\right)\int \frac{dx}{(x^2+a^2)^{n-1}} \quad (n>1),$$

which can be proven using integration by parts.

With $a = 2$ and $n = 3$ it yields

$$\int \frac{dx}{(x^2+4)^3} = \frac{x}{16(x^2+4)^2} + \frac{1}{4}\left(1 - \frac{1}{4}\right) \int \frac{dx}{(x^2+4)^2}.$$

Now taking $n = 2$ we have

$$= \frac{x}{16(x^2+4)^2} + \frac{3}{16}\left(\frac{x}{8(x^2+4)} + \frac{1}{4}\left(1 - \frac{1}{2}\right) \int \frac{dx}{x^2+4}\right)$$

$$= \frac{x}{16(x^2+4)^2} + \frac{3x}{128(x^2+4)} + \frac{3}{256} \arctan\left(\frac{x}{2}\right) + C,$$

in agreement with our previous result.