Improper Integrals of Type I

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Calculus II

Introduction

The Fundamental Theorem of Calculus tells us how to integrate any *continuous* function on a *closed interval* by using an antiderivative.

What if we wanted to integrate on more general sets such as *unbounded* intervals?

Or, what if we wanted to integrate a function with a *discontinuity*?

Integrals such as these are called *improper*, and we will define them using limits of "proper" integrals.

Motivating Example

Suppose we are asked to find the area of the region below $y = 1/(x+2)^2$, above the x-axis, and to the right of $x = 1$.

A natural expression for this area would be

$$\int_1^\infty \frac{dx}{(x+2)^2},$$

but the FTOC doesn't apply to such an integral, since $[1, \infty)$ is not a closed interval.

However, for any $t > 1$, FTOC *does* apply to

$$\int_1^t \frac{dx}{(x+2)^2} = \left.\frac{-1}{x+2}\right|_1^t = \frac{1}{3} - \frac{1}{t+2}.$$

As $t \to \infty$ we get more and more of the area under the curve.

So it is natural to define

$$\int_1^\infty \frac{dx}{(x+2)^2} = \lim_{t\to\infty} \int_1^t \frac{dx}{(x+2)^2} = \lim_{t\to\infty} \left(\frac{1}{3} - \frac{1}{t+2} \right) = \frac{1}{3}.$$

Note that even though the region in question is infinitely long, it has only a finite amount of area!

We use exactly the same procedure to integrate over any interval of the form $[a, \infty)$.

Improper Integrals of Type I

Definition (Improper Integrals of Type I)

If $f(x)$ is continuous on $[a, \infty)$, we define

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx,$$

provided the limit exists. In this case we say the improper integral is *convergent*. Otherwise it is *divergent*.

Remarks.

- Although $\int_a^b f(x)\,dx$ is defined for any continuous $f(x)$, $\int_a^\infty f(x)\,dx$ *may or may not* exist.
- We can define improper integrals of the form $\int_{-\infty}^b f(x)\,dx$ analogously.

Examples

Example 1

Evaluate $\int_0^\infty \frac{x}{1+x^2}\,dx$ or show that it diverges.

Solution. According to the definition we have

$$\begin{aligned}\int_0^\infty \frac{x}{1+x^2}\,dx &= \lim_{t\to\infty}\int_0^t \frac{x}{1+x^2}\,dx\\ &= \lim_{t\to\infty}\frac{1}{2}\ln(1+x^2)\Big|_0^t\\ &= \lim_{t\to\infty}\frac{1}{2}\ln(1+t^2)=\infty.\end{aligned}$$

Therefore the (improper) integral diverges. □

Example 2

Evaluate $\int_{-\infty}^{1} e^x \, dx$ or show that it diverges.

Solution. We have

$$\int_{-\infty}^{1} e^x \, dx = \lim_{t \to -\infty} \int_{t}^{1} e^x \, dx$$

$$= \lim_{t \to -\infty} e^x \Big|_{t}^{1}$$

$$= \lim_{t \to -\infty} e - e^t = \boxed{e}.$$

□

Example 3

Evaluate $\int_0^\infty (x^2 - 3x)e^{-x}\,dx$ or show that it diverges.

Solution. We use tabular integration by parts:

$u = x^2 - 3x$		$dv = e^{-x}\,dx$
$2x - 3$	$+$	$-e^{-x}$
2	$-$	e^{-x}
0	$+$	$-e^{-x}$

Thus

$$\int (x^2 - 3x)e^{-x}\,dx = -(x^2 - 3x)e^{-x} - (2x - 3)e^{-x} - 2e^{-x} + C$$

$$= -e^{-x}(x^2 - x - 1) + C.$$

Therefore

$$\int_0^\infty (x^2-3x)e^{-x}\,dx = \lim_{t\to\infty}\int_0^t (x^2-3x)e^{-x}\,dx$$
$$= \lim_{t\to\infty}\left(-e^{-x}(x^2-x-1)\Big|_0^t\right)$$
$$= \lim_{t\to\infty} -1 - e^{-t}(t^2-t-1)$$
$$= -1 - \lim_{t\to\infty}\frac{t^2-t-1}{e^t}$$
$$= -1 - \lim_{t\to\infty}\frac{2t-1}{e^t}$$
$$= -1 - \lim_{t\to\infty}\frac{2}{e^t} = \boxed{-1}$$

□

The following example will be particularly important when we study infinite series.

Example 4

For what values of $p > 0$ does $\displaystyle\int_1^\infty \frac{dx}{x^p}$ converge?

Solution. If $p = 1$ we have

$$\int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty} \int_1^t \frac{dx}{x} = \lim_{t\to\infty} \ln t = \infty,$$

so the integral diverges (to ∞).

If $p \neq 1$ we can use the power rule instead:

$$\int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty} \int_1^t x^{-p}\, dx = \lim_{t\to\infty} \frac{x^{1-p}}{1-p}\bigg|_1^t = \lim_{t\to\infty} \left(\frac{t^{1-p}}{1-p} + \frac{1}{p-1} \right).$$

Since

$$\lim_{t\to\infty} t^{\alpha} = \begin{cases} 0 & \text{if } \alpha < 0, \\ 1 & \text{if } \alpha = 0, \\ \infty & \text{if } \alpha > 0, \end{cases}$$

we find that the integral converges iff $1 - p < 0$ or $1 < p$, in which case the value is $\frac{1}{p-1}$.

To summarize:

$$\boxed{\int_1^\infty \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & \text{if } p > 1, \\ \infty & \text{if } 0 < p \leq 1. \end{cases}}$$

□

Remark. When we write $\displaystyle\int_a^\infty f(x)\,dx = \infty$ we mean that

$$\lim_{t\to\infty} \int_a^t f(x)\,dx = \infty.$$

In particular, we regard $\int_a^\infty f(x)\,dx$ as *divergent* in this case.

Integrals on $(-\infty, \infty)$

Given a continuous function $f(x)$ on $\mathbb{R} = (-\infty, \infty)$, we define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx,$$

provided *both* improper integrals converge (independently).

Example 5

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2 - 6x + 13}$ or show that it diverges.

Solution. First of all, we have

$$\int \frac{dx}{x^2 - 6x + 13} = \int \frac{dx}{(x-3)^2 + 4} = \frac{1}{2}\arctan\left(\frac{x-3}{2}\right) + C.$$

Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{x^2-6x+13} = \int_{-\infty}^{0} \frac{dx}{x^2-6x+13} + \int_{0}^{\infty} \frac{dx}{x^2-6x+13}$$

$$= \lim_{t\to-\infty} \frac{1}{2}\arctan\left(\frac{x-3}{2}\right)\Big|_t^0 + \lim_{s\to\infty} \frac{1}{2}\arctan\left(\frac{x-3}{2}\right)\Big|_0^s$$

$$= \lim_{t\to-\infty} \frac{1}{2}\left(\arctan\left(-\frac{3}{2}\right) - \arctan\left(\frac{t-3}{2}\right)\right)$$

$$+ \lim_{s\to\infty} \frac{1}{2}\left(\arctan\left(\frac{s-3}{2}\right) - \arctan\left(-\frac{3}{2}\right)\right)$$

$$= \frac{1}{2}\left(-\arctan\left(\frac{3}{2}\right) + \frac{\pi}{2}\right) + \frac{1}{2}\left(\frac{\pi}{2} + \arctan\left(\frac{3}{2}\right)\right) = \boxed{\frac{\pi}{2}}.$$

□

Example 6

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx$ or show that it does not exist.

Solution. Because

$$\int_0^{\infty} \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \int_0^{t} \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \frac{1}{2}\ln(x^2+1)\Big|_0^{\infty}$$

$$= \lim_{t\to\infty} \frac{1}{2}\ln(t^2+1) = \infty,$$

the integral $\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx$ does not exist. □

Principal Values of Improper Integrals

It is tempting to set

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_{-t}^{t} f(x)\,dx.$$

However, this can lead to counterintuitive results.
For instance, suppose we define

$$f(x) = \begin{cases} \frac{x}{x^2+1} & \text{if } x \geq 0, \\ \frac{4x}{4x^2+1} & \text{if } x < 0. \end{cases}$$

This is continuous on $\mathbb{R}$ and

$$\int_{-t}^{t} f(x)\,dx = \int_{-t}^{0} \frac{4x}{4x^2+1}\,dx + \int_{0}^{t} \frac{x}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln(4x^2+1)\Big|_{-t}^{0} + \frac{1}{2}\ln(x^2+1)\Big|_{0}^{t}$$

$$= -\frac{1}{2}\ln(4t^2+1) + \frac{1}{2}\ln(t^2+1)$$

$$= \frac{1}{2}\ln\frac{t^2+1}{4t^2+1}.$$

Thus

$$\lim_{t\to\infty}\int_{-t}^{t} f(x)\,dx = \lim_{t\to\infty}\frac{1}{2}\ln\frac{t^2+1}{4t^2+1} = -\ln 2.$$

However, we also have

$$\int_{-\infty}^{0} f(x)\,dx = \underbrace{\int_{-\infty}^{0}\frac{4x}{4x^2+1}\,dx}_{\text{sub. } u=-2x} = -\int_{0}^{\infty}\frac{u}{u^2+1} = -\int_{0}^{\infty} f(x)\,dx.$$

This would mean that

$$\int_{-\infty}^{\infty} f(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x) = 0 \neq -\ln 2.$$

The quantity obtained by letting the upper and lower limits tend to $\pm\infty$ *simultaneously* is nonetheless important, and is called the *Cauchy principal value* of the integral:

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_{-t}^{t} f(x)\,dx.$$

In the example above, $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ *does not exist* (why?), however

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = -\ln 2.$$

On the other hand, the sum law for limits (that exist!) implies:

Theorem 1

If $\int_{-\infty}^{\infty} f(x)\,dx$ *exists, then it equals P.V.* $\int_{-\infty}^{\infty} f(x)\,dx$.

However, this result is only useful if you can show $\int_{-\infty}^{\infty} f(x)\,dx$ exists *without* actually computing it.

Therefore, we will only occasionally be interested in Cauchy principal values of improper integrals over $\mathbb{R}$.