Linear Algebra
Spring 2021

Assignment 10
Due April 21

3.1. # 4, 10, 14

Exercise 1. Let $M_n(\mathbb{R})$ denote the vector space (actually an $\mathbb{R}$-*algebra*) of $n \times n$ matrices with entries in $\mathbb{R}$. Given $A \in M_n(\mathbb{R})$, let

$$H_A = \{B \in M_n(\mathbb{R}) \mid AB = BA\}.$$

Show that H_A is a subspace of $M_n(\mathbb{R})$.

Exercise 2. Given $A \in M_n(\mathbb{R})$, let

$$J_A = \{f(A) \mid f(X) \in \mathbb{R}[X]\} \subseteq M_n(\mathbb{R})$$

Show that J_A is a subspace of $M_n(\mathbb{R})$ by showing that

$$J_A = \mathrm{Span}\{I, A, A^2, A^3, \ldots\}.$$

Exercise 3. In the notation of the preceding two exercises, show that $J_A \leq H_A$.

Exercise 4. Let $H = \{f(X) \in \mathbb{R}[X] \mid f(0) = 0\}$.

a. Show that

$$H = X \cdot \mathbb{R}[X] = \{X \cdot f(X) \mid f(X) \in \mathbb{R}[X]\} = \mathrm{Span}\{X, X^2, X^3, \ldots\}.$$

b. Use the preceding part to show that H is a subspace of $\mathbb{R}[X]$. Find a basis for H.

c. What can you say about $H_a = \{f(X) \in \mathbb{R}[X] \mid f(a) = 0\}$, where $a \in \mathbb{R}$ is arbitrary?

Exercise 5. Let V denote the vector space of all sequences $\{a_n\}_{n=1}^{\infty}$ of real numbers. We will say that $\{a_n\}$ is *eventually zero* if there is an $N \in \mathbb{N}$ so that $a_n = 0$ for all $n \geq N$ (N can vary with the sequence). That is,

$$\{a_n\} = \{a_1, a_2, \ldots, a_{N-1}, 0, 0, 0, \ldots\}.$$

Let H denote the set of all eventually zero sequences. Show that H is a subspace of V. Find a basis of H.

Exercise 6. Let $a, b \in \mathbb{R}$ and set

$$H = \{e^{ax}(p(x)\cos bx + q(x)\sin bx) \,|\, p(x), q(x) \in \mathbb{R}[x]\}\,.$$

Show that H is a subspace of $C^1(\mathbb{R})$. Show that this conclusion remains valid if we replace $\mathbb{R}[x]$ in the definition of H with $\mathbb{P}_n$. Find a basis for H in both cases. [*Remark.* You'll need to know that the function $\tan x$ is *not* a rational function of x. Why is this true? Think in terms of vertical asymptotes.]

Exercise 7. Show that the set (field) of complex numbers $\mathbb{C}$ is an $\mathbb{R}$-vector space. Find an $\mathbb{R}$-basis for $\mathbb{C}$.