LINEAR ALGEBRA
SPRING 2021

ASSIGNMENT 11.2
DUE APRIL 21

Exercise 1. Let $a \in \mathbb{R}$ and define $E_a : \mathbb{R}[X] \to \mathbb{R}$ by $E_a(f(X)) = f(a)$. Determine $\ker E_a$ and $\operatorname{im} E_a$. Is E_a one-to-one? Onto?

Exercise 2. Given distinct $a_1, a_2, \ldots, a_n \in \mathbb{R}$, define $T : \mathbb{R}[X] \to \mathbb{R}^n$ by

$$T(f(X)) = \begin{pmatrix} f(a_1) \\ f(a_2) \\ \vdots \\ f(a_n) \end{pmatrix}.$$

It is not hard to show that T is a linear transformation.

- **a.** Determine $\ker T$ and $\operatorname{im} T$. Is T one-to-one? Onto?
- **b.** Suppose we replace the domain of T by $\mathbb{P}_m$. What happens to $\ker T$ and $\operatorname{im} T$? Your answer should depend on m and n.

Exercise 3. Let $\mathbb{R}^{\mathbb{N}}$ denote the vector space of all sequences of real numbers, and define the *shift operators* $L, R : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by

$$L(\{a_1, a_2, a_3, \ldots\}) = \{a_2, a_3, a_4, \ldots\} \quad \text{and} \quad R(\{a_1, a_2, a_3, \ldots\}) = \{0, a_1, a_2, a_3, \ldots\},$$

which we know to be linear transformations.

- **a.** Compute the kernel and image of L and R.
- **b.** Show that L is onto but not one-to-one, and that R is one-to-one but not onto. Conclude that $\mathbb{R}^{\mathbb{N}}$ *cannot* be finite dimensional.

Exercise 4. Let V and W be vector spaces and set

$$\mathcal{F}(V, W) = \{T : V \to W\},$$
$$\operatorname{Hom}(V, W) = \{T : V \to W \mid T \text{ is a linear transformation}\}.$$

Show that

$$\operatorname{Hom}(V, W) \leq \mathcal{F}(V, W),$$

where the operations in $\mathcal{F}(V, W)$ are pointwise.