Exercise 1. Let

$$A = \begin{pmatrix} -3 & 1 \\ 2 & 4 \end{pmatrix}.$$

Use the linear transformation $B \mapsto AB - BA$ to find a basis for the subspace

$$H_A = \{B \in M_2(\mathbb{R}) \mid AB = BA\}$$

of $M_2(\mathbb{R})$, as we did in class. Is it true that

$$H_A = \operatorname{Span}\{I, A, A^2, A^3, \ldots\}?$$

Exercise 2. For $a, b \in \mathbb{R}$ with $b \neq 0$, let

$$V_n(a,b) = \{e^{ax}(p(x)\cos bx + q(x)\sin bx) \mid p(x), q(x) \in \mathbb{P}_n\},$$

which we know to be a subspace of $C^1(\mathbb{R})$.

a. If $f(x) = e^{ax}(p(x)\cos bx + q(x)\sin bx) \in V_n(a,b)$, show that

$$\frac{df}{dx} = e^{ax}((a \cdot p(x) + p'(x) + b \cdot q(x))\cos bx + (a \cdot q(x) + q'(x) - b \cdot p(x))\sin bx).$$

Conclude that $\frac{d}{dx}$ is an *endomorphism* of $V_n(a,b)$, i.e. is a linear transformation whose domain and codomain are both equal to $V_n(a,b)$.

b. We have seen that

$$\mathcal{B} = \{e^{ax}x^j \cos bx \mid 0 \leq j \leq n\} \cup \{e^{ax}x^j \sin bx \mid 0 \leq j \leq n\}$$

is a basis for $V_n(a,b)$. Order $\mathcal{B}$ as follows:

$$e^{ax}\cos bx, e^{ax}\sin bx, e^{ax}x\cos bx, e^{ax}x\sin bx, \ldots, e^{ax}x^j\cos bx, e^{ax}x^j\sin bx, \ldots$$

Compute the matrix for $\frac{d}{dx}$ relative to $\mathcal{B}$ (in this order).

c. Use the result of part **b** to conclude that $\frac{d}{dx}$ is invertible on $V_n(a,b)$. Explain why this proves that every $f(x) \in V_n(a,b)$ has a *unique* antiderivative in $V_n(a,b)$.

d. Invert the matrix you found in part **b** in the case $n = 1$. Use this to compute

$$\int e^{ax}((x+1)\cos bx + (2x-1)\sin bx)\,dx$$

via matrix multiplication.