Exercise 1. Let $\mathbb{R}^{\mathbb{N}}$ denote the vector space of all sequences with real entries, let $a, b \in \mathbb{R}$, and define $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by

$$T(\{x_n\}_{n=1}^{\infty}) = \{x_{n+2} - ax_{n+1} - bx_n\}_{n=1}^{\infty} = (L^2 - aL - bI)(\{x_n\}_{n=1}^{\infty}),$$

where L is the left shift operator defined previously.

a. Show that T is a linear transformation.

b. Show that $\{x_n\} \in \ker T$ if and only if $x_{n+2} = ax_{n+1} + bx_n$ for all $n \geq 1$.

c. Show that $\dim \ker T = 2$. [*Suggestion.* Show that any sequence satisfying the conditions of part **b** is completely determined by its first two terms.]

Exercise 2. Let T be as in the preceding exercise. The *characteristic polynomial* of T is

$$f(X) = X^2 - aX - b.$$

a. If $r \in \mathbb{R}$ is a root of $f(X)$, show that $\{r^{n-1}\}_{n=1}^{\infty} \in \ker T$. [*Suggestion.* Multiply the relationship $f(r) = 0$ by r^n and use part **b** above.]

b. If r_1 and r_2 are distinct real numbers, show that the sequences $\{r_1^{n-1}\}$ and $\{r_2^{n-1}\}$ are linearly independent.

c. If $r_1 \neq r_2$ are both real roots of $f(X)$, use part **c** above to show that

$$\ker T = \operatorname{Span}\{\{r_1^{n-1}\}, \{r_2^{n-1}\}\}.$$

Exercise 3. The *Fibonacci numbers* are the terms of the sequence $\{F_n\}$ defined by $F_1 = 1$, $F_2 = 1$, and

$$F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geq 1.$$

a. Use the result of part **c** above to find constants $a_1, a_2 \in \mathbb{R}$ so that

$$F_n = a_1 r_1^{n-1} + a_2 r_2^{n-1} \quad \text{for all } n \geq 1,$$

where r_1, r_2 are the roots of the characteristic polynomial

$$f(X) = X^2 - X - 1.$$

b. Derive the formula

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$