Exercise 1. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \left\{ \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Compute $[I]_{\mathcal{B}}^{\mathcal{C}}$ and $[I]_{\mathcal{C}}^{\mathcal{B}}$ by hand without using row reduction.

Exercise 2. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \\ -2 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \left\{ \begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} \right\}.$$

Compute $[I]_{\mathcal{B}}^{\mathcal{C}}$ and $[I]_{\mathcal{C}}^{\mathcal{B}}$.

Exercise 3. Let

$$A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix},$$

and define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. Use change of basis matrices to compute the matrix for T relative to the basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix} \right\}.$$

[*Suggestion:* You can speed up your computations by setting $\alpha = 1+\sqrt{2}$, $\beta = 1-\sqrt{2}$, and then noting that

$$(X-\alpha)(X-\beta) = X^2 - 2X - 1.$$

This gives several useful algebraic relationships between α and β.]

Exercise 4. Let $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be (counterclockwise) rotation about the origin by θ radians. What happens if you compute the matrix for T_θ relative to the *complex* basis[1]

$$\mathcal{C} = \left\{ \begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\}?$$

[*Suggestion.* Remember *Euler's formula*, $e^{i\theta} = \cos\theta + i\sin\theta$.]

[1] We can extend any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ to a linear transformation $\widehat{T} : \mathbb{C}^n \to \mathbb{C}^m$ of *complex* vector spaces by simply viewing the standard matrix for T as an element of $\mathrm{M}_{m\times n}(\mathbb{C})$.