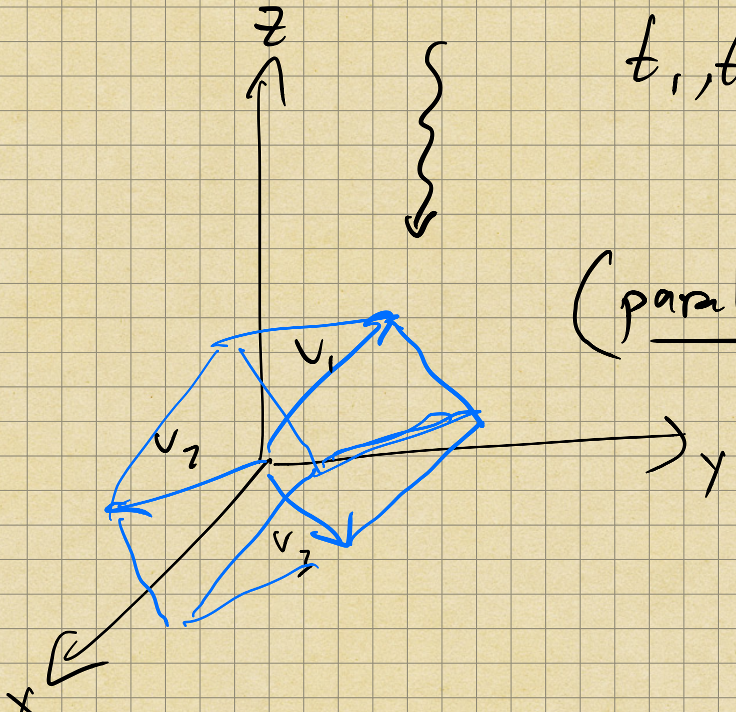
Intro. to Determinants

Given $v_1, v_2, v_3 \in \mathbb{R}^3$, let

$$\langle v_1, v_2, v_3 \rangle = \{ t_1 v_1 + t_2 v_2 + t_3 v_3 \mid t_1, t_2, t_3 \in [0,1] \}$$

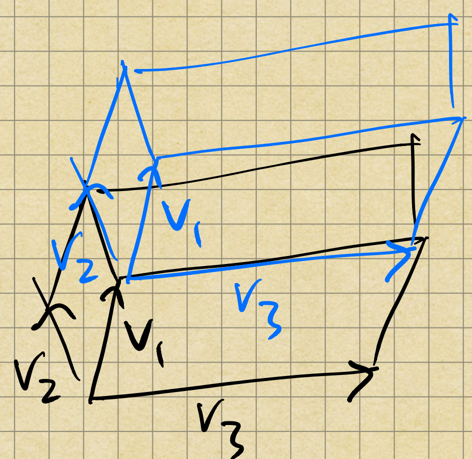


(parallelepiped)

$$\text{Vol.} \langle v_1, v_2, v_3 \rangle = ?$$

We will find a formula for $\text{Vol.}\langle v_1, v_2, v_3 \rangle$ by studying its properties as a fn. $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ (a functional or form)

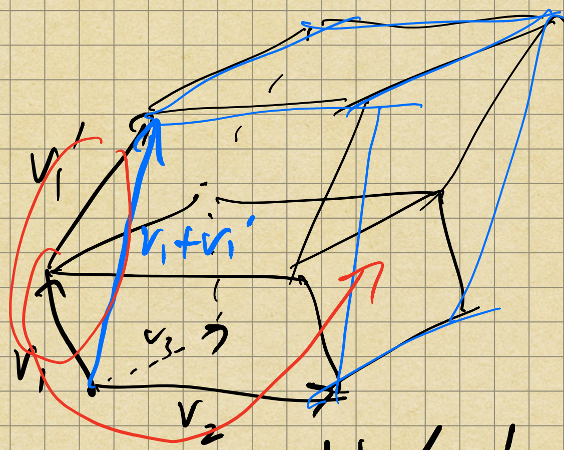
Properties of $\text{Vol}\langle v_1, v_2, v_3\rangle$:



$$\text{Vol}\langle 2v_1, v_2, v_3\rangle$$
$$= 2\cdot\text{Vol}\langle v_1, v_2, v_3\rangle$$

Preserves scalar mult. in each variable.

$$\boxed{\begin{aligned}\text{Vol}\langle cv_1, v_2, v_3\rangle &= \text{Vol}\langle v_1, cv_2, v_3\rangle \\ &= \text{Vol}\langle v_1, v_2, cv_3\rangle \\ &= c\cdot\text{Vol}\langle v_1, v_2, v_3\rangle\end{aligned}}$$



$$\text{Vol.}\langle v_1, v_2, v_3\rangle + \text{Vol.}\langle v_1', v_2, v_3\rangle$$

$$= \text{Vol.}\langle v_1 + v_1', v_2, v_3\rangle$$

$\text{Vol.}\langle v_1, v_2, v_3\rangle$ respects vector addition in each variable

Moral: $\text{Vol}\langle v_1, v_2, v_3\rangle$ is multilinear

$$\text{Vol.}\langle v_1, v_1, v_2\rangle = 0$$

↳ If any two vectors are the same, Vol. $= 0$.

$$\mathrm{Vol}\langle v_1+v_2, v_1+v_2, v_3\rangle = \underline{\underline{0}}$$

$$\| $$

$$\cancel{\mathrm{Vol.}\langle v_1, v_1, v_3\rangle}^{\,0} + \underline{\mathrm{Vol.}\langle v_1, v_2, v_3\rangle}$$
$$+ \underline{\mathrm{Vol.}\langle v_2, v_1, v_3\rangle} + \cancel{\mathrm{Vol.}\langle v_2, v_2, v_3\rangle}^{\,0}$$

$$\Rightarrow \quad \mathrm{Vol.}\langle v_2, v_1, v_3\rangle = -\,\mathrm{Vol.}\langle v_1, v_2, v_3\rangle$$

Interchanging any two vectors in $\mathrm{Vol}\langle v_1, v_2, v_3\rangle$ simply negates the value

↳ i.e. $\mathrm{Vol.}\langle v_1, v_2, v_3\rangle$ is <u>alternating</u>.

We must choose the "unit volume":

$$\boxed{\mathrm{Vol.}\langle e_1, e_2, e_3\rangle = 1}$$

Conclusion: $\mathrm{Vol.}\langle v_1, v_2, v_3\rangle$ is the unique multilinear, alternating functional w/ $\mathrm{Vol.}\langle e_1, e_2, e_3\rangle = 1$.

Proof of Uniqueness:

$$\mathrm{Vol.}\langle v_1, v_2, v_3\rangle$$

$$= \mathrm{Vol.}\left\langle \begin{pmatrix} a \\ b \\ c \end{pmatrix}, v_2, v_3 \right\rangle$$

$$= \mathrm{Vol.}\langle ae_1 + be_2 + ce_3, v_2, v_3\rangle$$

$$= a\,\mathrm{Vol.}\langle e_1, v_2, v_3\rangle + b\,\mathrm{Vol}\langle e_2, v_2, v_3\rangle + c\,\mathrm{Vol.}\langle e_3, v_2, v_3\rangle$$

$$\begin{pmatrix} r \\ s \\ t \end{pmatrix} = re_1 + se_2 + te_3$$

$$= a\left(s \operatorname{Vol.}\langle e_1, e_2, v_3 \rangle + t \operatorname{Vol}\langle e_1, e_3, v_3 \rangle \right)$$

$$+ b\left(r \operatorname{Vol.}\langle e_2, e_1, v_3 \rangle + t \operatorname{Vol}\langle e_2, e_3, v_3 \rangle \right)$$

$$+ c\left(r \operatorname{Vol.}\langle e_3, e_1, v_3 \rangle + s \operatorname{Vol}\langle e_3, e_2, v_3 \rangle \right)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x e_1 + y e_2 + z e_3$$

$$= a\left(s z \underbrace{\operatorname{Vol.}\langle e_1, e_2, e_3 \rangle}_{1} + t y \underbrace{\operatorname{Vol.}\langle e_1, e_3, e_2 \rangle}_{-1} \right)$$

$$+ b\left(r z \underbrace{\operatorname{Vol.}\langle e_2, e_1, e_3 \rangle}_{-1} + t x \underbrace{\operatorname{Vol}\langle e_2, e_3, e_1 \rangle}_{1} \right)$$

$$+ c\left(r y \underbrace{\operatorname{Vol.}\langle e_3, e_1, e_2 \rangle}_{1} + s x \underbrace{\operatorname{Vol.}\langle e_3, e_2, e_1 \rangle}_{-1} \right)$$

$$= \boxed{asz - aty - brz + btx + cry - csx}$$

$$= \mathrm{Vol.}\langle v_1, v_2, v_3 \rangle$$

$$= \boxed{\begin{vmatrix} a & r & x \\ b & s & y \\ c & t & z \end{vmatrix}}$$

determinant

Notice:

$$\left|\begin{matrix} a & r & x \\ b & s & y \\ c & t & z \end{matrix}\right| \begin{matrix} a & r \\ b & s \\ c & t \end{matrix}$$

$$= asz + ryc + xbt$$
$$- xsc - ayt - rbz$$

Warning: This <u>does not</u> compute larger determinants.

Claim:

$$\mathrm{Vol.}\langle v_1, v_2, v_3 \rangle = \begin{vmatrix} v_1 & v_2 & v_3 \end{vmatrix}$$

$= \det(v_1\ v_2\ v_3)$

$= 0 \iff \{v_1, v_2, v_3\}$ is lin. dep.

Idea: $\{v_1, v_2, v_3\}$ lin. dep.

(WLOG) $\Rightarrow v_3 = c_1 v_1 + c_2 v_2$

$$\Rightarrow \begin{vmatrix} v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} v_1 & v_2 & c_1 v_1 + c_2 v_2 \end{vmatrix}$$

$$= c_1 \begin{vmatrix} v_1 & v_2 & v_1 \end{vmatrix} + c_2 \begin{vmatrix} v_1 & v_2 & v_2 \end{vmatrix}$$

$$= 0 + 0 = 0$$

$\underbrace{\{v_1, v_2, v_3\} \text{ lin. ind.}}_{\text{in } \mathbb{R}^3 \text{//}} \Rightarrow \operatorname{Span}\{v_1, v_2, v_3\} = \mathbb{R}^3$

So $\text{Vol.}\langle v_1, v_2, v_3 \rangle \neq 0$,

because parallelepiped cannot be "flat."

Final Remark: Similar considerations show that

$$ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \text{Area}\underbrace{\left\langle \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \right\rangle}_{\text{parallelogram}}$$

is the unique multilinear, alternating form w/ $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$.