# Determinants

Let $M_n(\mathbb{R}) = \{ n \times n \text{ matrices w/ real entries} \}$

A function $D: M_n(\mathbb{R}) \to \mathbb{R}$ is called:

- **multilinear** if

$$D(a_1, a_2, \dots, ca_i + da_i', \dots, a_n)$$
$$= cD(a_1, a_2, \dots, a_i, \dots, a_n) + dD(a_1, a_2, \dots, a_i', \dots, a_n)$$

for all $i = 1, 2, \dots, n$

($D$ is linear in each column)

- **alternating** if

$$D(a_1, \dots, a_i, \dots, a_j, \dots, a_n) = -D(a_1, \dots, a_j, \dots, a_i, \dots, a_n)$$

$$(\Rightarrow D(\dots, a, \dots, a, \dots) = 0)$$

**Theorem**: For any scalar $a \in \mathbb{R}$, there exists a unique multilinear alternating $D: M_n(\mathbb{R}) \to \mathbb{R}$ w/ $D(I) = a$.

When $a=1$, we write $D = \det$ (the determinant). For general $a$, we have $D(A) = a \cdot \det(A)$.

**Proof (Sketch)**: If $A = (a_{ij})$, then

$$A = \left( \sum_{i_1=1}^{n} a_{i_1 1} e_{i_1} \quad \sum_{i_2=1}^{n} a_{i_2 2} e_{i_2} \cdots \cdots \sum_{i_n=1}^{n} a_{i_n n} e_{i_n} \right)$$

to expand $D(A)$, take one summand from each col. in every possible way

$$D(A) = \sum_{(i_1, i_2, \dots, i_n)} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n} D(e_{i_1}, \dots, e_{i_n})$$

(multilinear)

$\hookrightarrow$ in $\{1, 2, \dots, n\}$

If $D$ is alternating:

$$D(e_{i_1},\dots,e_{i_n}) = 0$$

if any two $e_i$ are the same. So $(i_1,\dots,i_n)$ need only run over <u>permutations</u> of $\{1,2,\dots,n\}$:

$$S_n = \{\text{permutations of } \{1,2,\dots,n\}\}$$

$$= \sum_{\sigma\in S_n} a_{\sigma(1)1}\, a_{\sigma(2)2}\cdots a_{\sigma(n)n}\, D(e_{\sigma(1)},\dots,e_{\sigma(n)})$$

(swap cols to get $I$)

$$(*) \quad = \underbrace{\left(\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, a_{\sigma(1)1}\cdots a_{\sigma(n)n}\right)}_{\det A} D(I)$$

This proves existence and uniqueness of $D$. $\blacksquare$

**Cor:** $\det A = \det A^T$

**Proof:** (Sketch) Replace $\sigma$ w/ $\sigma^{-1}$ in (*) above. ☒

**Cor:** $\det A$ is multilinear/alternating in both cols. **and** rows.

**Cor:** $\det(AB) = \det A \det B$

**Proof:** The rule

$$B \overset{D}{\longmapsto} \det(AB)$$

is multilinear/alternating (in cols. of B).

By the Thm.:

$$\begin{aligned} D(B) &= \det(B) \cdot D(I) \\ &= \det(B) \det(A \cdot I) \\ &= \det(A) \det(B) \end{aligned}$$

☒

## Computing det(A): We start small:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= \left| a\begin{pmatrix}1\\0\end{pmatrix} + c\begin{pmatrix}0\\1\end{pmatrix} \quad \begin{matrix} b \\ d \end{matrix} \right|$$

$$= a\begin{vmatrix} 1 & b \\ 0 & d \end{vmatrix} + c\begin{vmatrix} 0 & b \\ 1 & d \end{vmatrix}$$

$$= a\left| \begin{matrix}1\\0\end{matrix} \quad b\begin{pmatrix}1\\0\end{pmatrix} + d\begin{pmatrix}0\\1\end{pmatrix} \right| + c\left| \begin{matrix}0\\1\end{matrix} \quad b\begin{pmatrix}1\\0\end{pmatrix} + d\begin{pmatrix}0\\1\end{pmatrix} \right|$$

$$= a\left( b\,\cancel{\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}}^{\,0} + d\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right) \to 1$$

requires swap

$$+ c\left( b\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + d\,\cancel{\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix}}^{\,0} \right)$$

$$= \underline{\underline{ad - bc}}$$