Computing Determinants

<u>Theorem</u>: There is a unique multilinear, alternating form

$$\det : M_n(\mathbb{R}) \to \mathbb{R}$$

w/ $\det I = 1$.

<u>Remark</u>: Because $\det A = \det A^T$, $\det A$ is multilinear/alternating in both rows <u>and</u> cols.

Determinants and Row/Col. Ops

a) If B is obtained from A by a single replacement, then

$$\det A = \det B$$

→ b) If B is obtained from A by a single interchange, then

$$\det A = -\det B$$

→ c) If B is obtained from A by scaling a single row/col. by c, then

$$\det B = c \det A$$

Proof of (a):

$$\left[\det(\cdots \vec{a} \xrightarrow{\times c} \vec{b} \cdots)\right]$$

$$\det(\cdots \vec{a} \cdots \vec{b}+c\vec{a} \cdots)$$

$$= \det(\cdots \vec{a} \cdots \vec{b} \cdots) + c\,\underbrace{\cancel{\det(\cdots \vec{a} \cdots \vec{a} \cdots)}}_{0}$$

$$= \det(\cdots \vec{a} \cdots \vec{b} \cdots) \quad \blacksquare$$

Cofactor Expansions

$$\det A = \begin{vmatrix} a_1 & r_1^T \\ a_2 & r_2^T \\ \vdots & \vdots \\ a_n & r_n^T \end{vmatrix}$$

$$
= \begin{vmatrix} & r_1^T \\ a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n & r_2^T \\ & \vdots \\ & r_n^T \end{vmatrix}
$$

$$
= a_1 \begin{vmatrix} 1 & r_1^T \\ 0 & r_2^T \\ \vdots & \vdots \\ 0 & r_n^T \end{vmatrix} + a_2 \begin{vmatrix} 0 & r_1^T \\ 1 & r_2^T \\ 0 & r_3^T \\ \vdots & \vdots \\ 0 & r_n^T \end{vmatrix}
$$

$$
\cdots + a_n \begin{vmatrix} 0 & r_1^T \\ \vdots & \vdots \\ 0 & \\ 1 & r_n^T \end{vmatrix}
$$

$$
= a_1 \begin{vmatrix} 1 & 0 \cdots & 0 \\ 0 & r_2^T & \\ \vdots & \vdots & \\ 0 & r_n^T & \end{vmatrix}
$$

$(n-1)\times(n-1)$

A_{11}

$$
\begin{vmatrix} 1 & 0 & \cdots & 0 \end{vmatrix}
$$

$$
-a_2 \begin{vmatrix} 0 & r_1^T \\ 0 & r_3^T \\ \vdots & \vdots \\ 0 & r_n^T \end{vmatrix} \quad A_{21}
$$

$$
\cdots + a_n \begin{vmatrix} 0 & r_1^T \\ \vdots & r_2^T \\ & \vdots \\ 1 & 0 \cdots 0 \end{vmatrix}
$$

$$
\begin{aligned}
= a_1 \det A_{11} &- a_2 \det A_{21} \\
&+ a_3 \det A_{31} \cdots + (-1)^i a_i \det A_{i1} \\
&\cdots + (-1)^n a_n \det A_{n1}.
\end{aligned}
$$

<u>Remarks</u>:

1. We can expand in this way along <u>any</u> row or col. of A. We get

$$\det A = \sum_{i=1}^{n} a_{ij} (-1)^{i+j} \det A_{ij}$$

down j^{th} col.

A_{ij} = matrix obtained by deleting i^{th} row + j^{th} col. from A.

And similarly for row expansions.

2. The (i,j) <u>cofactor</u> of A is

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

<u>Ex</u>:

$$\begin{vmatrix} -1 & 3 & 2 & 0 \\ 8 & 3 & 1 & -2 \\ 0 & -1 & 3 & 1 \\ 0 & 9 & 1 & 4 \end{vmatrix}$$

Remark: The signs in any cofactor expansion obey:

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & & & & \\ \vdots & & & & \end{pmatrix}$$

$$= (-1) \begin{vmatrix} 3 & 1 & -2 \\ -1 & 3 & 1 \\ 9 & 1 & 4 \end{vmatrix}$$

$$-8\begin{vmatrix} 3 & 2 & 0 \\ -1 & 3 & 1 \\ 9 & 1 & 4 \end{vmatrix}$$

$$+\,0\cdot\left|\ \sim\!\sim\ \right|$$

$$-\,0\cdot\left|\ \sim\!\sim\ \right|$$

$$= -\left(-(-1)\begin{vmatrix} 1 & -2 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 3 & -2 \\ 9 & 4 \end{vmatrix} - \begin{vmatrix} 3 & 1 \\ 9 & 1 \end{vmatrix}\right)$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$$

$$-8\left(-1\cdot\begin{vmatrix} 3 & 2 \\ 9 & 1 \end{vmatrix} + 4\begin{vmatrix} 3 & 2 \\ -1 & 3 \end{vmatrix}\right)$$

$$= -(4+2) - 3(12+18)$$

$$+ (3-9) + 8(3-18)$$

$$- 32(9+2)$$

$$= -6 - 90 - 6 - 120$$

$$- 320 - 32$$

$$= -102 - 120 - 352$$

$$= \boxed{-574}$$

<u>Evaluating det A via</u>

<u>Row Reduction</u>

Lemma :

Upper triangular matrix

$$\begin{vmatrix} d_1 & * & * & \cdots \\ & d_2 & * & * \\ & 0 & \ddots & \\ & & & d_n \end{vmatrix} = d_1 d_2 \cdots d_n$$

Proof :

$$\begin{vmatrix} d_1 & & * \\ & \ddots & \\ 0 & & d_n \end{vmatrix} = d_1 \begin{vmatrix} d_2 & & * \\ & \ddots & \\ & & d_n \end{vmatrix}$$

$$= d_1 d_2 \begin{vmatrix} d_3 & & * \\ & \ddots & \\ & & d_n \end{vmatrix}$$

$$\vdots$$

$$= d_1 d_2 \cdots d_n \qquad \blacksquare$$

$\underline{Ex}$: $\begin{vmatrix} -1 & 3 & 2 & 0 \\ 8 & 3 & 1 & -2 \\ 0 & -1 & 3 & 1 \\ 0 & 9 & 1 & 4 \end{vmatrix}$

$$= \begin{vmatrix} -1 & 3 & 2 & 0 \\ 0 & 27 & 17 & -2 \\ 0 & -1 & 3 & 1 \\ 0 & 9 & 1 & 4 \end{vmatrix}$$

$$= - \begin{vmatrix} -1 & 3 & 2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 27 & 17 & -2 \\ 0 & 9 & 1 & 4 \end{vmatrix}$$

$$= - \begin{vmatrix} -1 & 3 & 2 & 0 \\ 0 & -1 & 3 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 0 & 98 & 25 \\ 0 & 0 & 28 & 13 \end{vmatrix}$$

$$= -(-1)\begin{vmatrix} -1 & 3 & 1 \\ 0 & 98 & 25 \\ 0 & 28 & 13 \end{vmatrix}$$

$$= -(-1)(-1)\begin{vmatrix} 98 & 25 \\ 28 & 13 \end{vmatrix}$$

$$= -(98\cdot 13 - 25\cdot 28)$$

$$= -574 \quad \checkmark$$

Ex:

$$\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 4 & 0 & 3 & -5 \\ \cancel{7} & \cancel{3} & \cancel{4} & \cancel{-8} \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$

$$= -2(3) \begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= -6 \begin{vmatrix} 4 & 3 & 1 \\ 5 & 2 & 1 \\ 0 & -1 & 0 \end{vmatrix}$$

$$= -6 - (-1) \begin{vmatrix} 4 & 1 \\ 5 & 1 \end{vmatrix}$$

$$= -6 + (4 - 5) = -7$$

Remark: Given a matrix A, its ***classical adjoint*** is the matrix

$$B = \left((-1)^{i+j} A_{ji} \right)$$

Note that the ij entry of AB is

$$\sum_k a_{ik} (-1)^{k+j} A_{jk}$$

This is the cofactor expansion of:

$$\begin{array}{r} \\ j^{th} \text{ row} \to \\ \\ \end{array} \begin{vmatrix} (\text{rest of } A) \\ a_{i1} \quad a_{i2} \cdots a_{in} \\ (\text{rest of } A) \end{vmatrix} = \begin{cases} \det A, & i=j \\ 0, & i \neq j \end{cases}$$

Since in this case the row $(a_{i1}\ a_{i2} \cdots a_{in})$ occurs in both the i^{th} <u>and</u> j^{th} rows.

Thus:

$$AB = \begin{pmatrix} \det A & & \\ & \ddots & \\ & & \det A \end{pmatrix}$$

$$= \det A \cdot I$$

If $\det A \neq 0$, this proves

$$A^{-1} = \frac{1}{\det A}\left((-1)^{i+j} C_{ji} \right)$$