Determinants & Invertibility

If A is invertible:

$$AB = I \Rightarrow \det(AB) = \det I$$

$$\Rightarrow \det A \cdot \det B = 1$$

$$\Rightarrow \det A \neq 0$$

$$\boxed{A \text{ invertible} \Rightarrow \det A \neq 0}$$

Goal: Prove converse statement

A $(n \times n)$ is arbitrary,

$\vec{b} \in \mathbb{R}^n$, consider

$$A\vec{x} = \vec{b}$$

Introduce new variable z and consider homog. system

$$A\vec{x} - z\vec{b} = \vec{0}$$

Claim:

Remark: We are essentially moving from an eqn. in Euclidean space ($\mathbb{R}^n$) to an eqn. in projective space ($\mathbb{P}^n$)

$A\vec{x} = \vec{b}$ has a sol'n

$$\Updownarrow$$

$A\vec{x} - z\vec{b} = \vec{0}$ has a sol'n w/ $z \neq 0$

Proof: ($\Rightarrow$) Let $z=1$

($\Leftarrow$) Divide through by $z \neq 0$. $\blacksquare$

Why? We can solve

$$A\vec{x} - z\vec{b} = \vec{0}$$

using determinants!

Cross Product:

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Write $A = (\vec{a}_1, \vec{a}_2 \cdots \vec{a}_n)$ and consider

Remark: If we replace $\alpha_1, \ldots, \alpha_{n+1}$ w/ $e_1, e_2, \ldots, e_{n+1}$, this is just an n-dim. "cross product."

$$\det \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n & \alpha_{n+1} \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n & -\vec{b} \end{pmatrix}$$

$A \quad -\vec{b}$

Cofactor exp. along top row

$$= \alpha_1 \begin{vmatrix} \vec{a}_2 & \cdots & \vec{a}_n & -\vec{b} \end{vmatrix}$$

$$- \alpha_2 \begin{vmatrix} \vec{a}_1 & \vec{a}_3 & \cdots & \vec{a}_n & -\vec{b} \end{vmatrix}$$

$$\vdots$$

$$(-1)^{n+1} \begin{vmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{n-1} & -\vec{b} \end{vmatrix} \alpha_n$$

$$+ (-1)^{n+2} \begin{vmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{vmatrix} \alpha_{n+1}$$

If $(\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n \ \alpha_{n+1})$

is <u>any</u> row of $(A \ -\vec{b})$,

this det. $= 0$. That is

$$\begin{array}{c} x_1 \rightarrow \\ x_2 \rightarrow \\ \vdots \\ x_n \rightarrow \\ z \rightarrow \end{array} \begin{pmatrix} |\vec{a}_2 \cdots \vec{a}_n \ -\vec{b}| \\ -|\vec{a}_1 \ \vec{a}_3 \cdots \vec{a}_n \ -\vec{b}| \\ \vdots \\ (-1)^{n+1} |\vec{a}_1 \cdots \vec{a}_{n-1} \ -\vec{b}| \\ (-1)^{n+2} |\vec{a}_1 \cdots \vec{a}_n| \end{pmatrix}$$

is a sol'n to $A\vec{x} - z\vec{b} = \vec{0}$!

det A!

Moral: If $\det A \neq 0$, we can divide by z to get a sol'n to $A\vec{x} = \vec{b}$.

When we do this and perform some convenient column swaps, we find that:

$$(*) \quad \boxed{\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad x_i = \frac{|\vec{a}_1 \cdots \overset{\text{ith col.}}{\vec{b}} \cdots \vec{a}_n|}{\det A}}$$

solves $A\vec{x} = \vec{b}$!

Cramer's Rule: If $\det A \neq 0$, then $(*)$ solves $A\vec{x} = \vec{b}$.

Cor.: If $\det A \neq 0$, then A is invertible.

Proof: By Cramer's Rule, $A\vec{x} = \vec{b}$ has a sol'n for every $\vec{b}$. $\blacksquare$

Invertible Matrix Thm. (Cont.)

For a square matrix A, TFAE:

(a) A is invertible

$\vdots$

(x) $\det A \neq 0$.

Ex: Solve

$$\begin{aligned} x_1 + x_2 \phantom{{}+2x_3} &= 3 \\ -3x_1 \phantom{{}+x_2} + 2x_3 &= 0 \\ x_2 - 2x_3 &= 2 \end{aligned}$$

using Cramer's Rule.

Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -3 & 0 & 2 \\ 0 & 1 & -2 \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$$

Then

$$\det A = \left|\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & 1 \\ -3 & 0 & 2 & -3 & 0 \\ 0 & 1 & -2 & 0 & 1 \end{array}\right|$$

$$= -2 - 6 = -8$$

$$\left|\vec{b} \quad \vec{a}_2 \quad \vec{a}_3\right| = \begin{vmatrix} 3 & 1 & 0 \\ 0 & 0 & 2 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix}$$

$$= -2$$

$$\left|\vec{a}_1 \quad \vec{b} \quad \vec{a}_3\right| = \left|\begin{array}{ccc|cc} 1 & 3 & 0 & 1 & 3 \\ -3 & 0 & 2 & -3 & 0 \\ 0 & 2 & -2 & 0 & 2 \end{array}\right|$$

$$= -4 - 18 = -22$$

$$|\vec{a}_1 \ \vec{a}_2 \ \vec{b}| = \begin{vmatrix} 1 & 1 & 3 \\ -3 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= -(-3) \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= -3$$

So the sol'n is

$$\boxed{\vec{x} = \begin{pmatrix} 1/4 \\ 11/24 \\ 3/8 \end{pmatrix}}$$

∎

Cramer's Rule + Matrix Inversion

Suppose $\det A \neq 0$, so A is invertible. To find $A^{-1} = B$ we need to solve

$$AB = I$$

for B. Write

$$B = (b_1 \; b_2 \; \cdots \; b_n)$$
$$I = (e_1 \; e_2 \; \cdots \; e_n)$$

Then

$$AB = I \iff$$
$$(Ab_1 \; Ab_2 \; \cdots \; Ab_n) = (e_1 \; \cdots \; e_n)$$

$\Leftarrow$

$$Ab_j = e_j$$

i.e. j^{th} col. of $B = A^{-1}$ is the

soln to $Ax = e_j$. So

the ij-entry of A^{-1} is

given by

$$\frac{\begin{vmatrix} a_1 & \cdots & \overset{i^{th}\text{ col.}}{e_j} & \cdots & a_n \end{vmatrix}}{\det A}$$

Cofactor expand down i^{th} col.

$$= \frac{(-1)^{i+j} \det A_{ji}}{\det A}$$

ji cofactor of A!

Theorem: If $\det A \neq 0$, then

$$A^{-1} = \frac{1}{\det A} \boxed{\left((-1)^{i+j} \det A_{ji} \right)}$$

<u>Classical Adjoint of A</u>: adj A

<u>Ex</u>: Use classical adjoint to invert

$$A = \begin{pmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

We have

$$\det A = \begin{vmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix} \begin{matrix} 3 & 5 \\ 1 & 0 \\ 2 & 1 \end{matrix}$$

$= 10 + 4 - 3 - 5 = 6$

The cofactors are:

$$\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1, \quad -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1$$

$$\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$$

$$-\begin{vmatrix} 5 & 4 \\ 1 & 1 \end{vmatrix} = -1, \quad \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = -5$$

$$-\begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} = 7$$

$$\begin{vmatrix} 5 & 4 \\ 0 & 1 \end{vmatrix} = 5, \quad -\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} = 1,$$

$$\begin{vmatrix} 3 & 5 \\ 1 & 0 \end{vmatrix} = -5$$

$$\frac{1}{6}\underbrace{\begin{pmatrix} -1 & 1 & 1 \\ -1 & -5 & 7 \\ 5 & 1 & -5 \end{pmatrix}^{T}}_{\text{Adj } A}$$

$$\Downarrow$$

$$\boxed{A^{-1} = \frac{1}{6}\begin{pmatrix} -1 & -1 & 5 \\ 1 & -5 & 1 \\ 1 & 7 & -5 \end{pmatrix}}$$

$$\blacksquare$$