Lin. Alg. Exam 2 :

When : Monday (4/5)

What : Everything through determinants...

1.9 (Matrices of Linear Trans.)

2.1 - 2.3 (Matrix Algebra/Inversion)

3.1 - 3.3 (Determinants)

HW : Forthcoming...

Vector Spaces

A vector space is a set V of "vectors" together w/ two binary operations

#1 ⟶ $+ : V \times V \longrightarrow V$ (addition)

$(v, w) \longmapsto v + w$

#6 → $\cdot : \mathbb{R} \times V \longrightarrow V$ (scalar mult.)

$(c, v) \longmapsto cv$

so that:

(V, +) is a group

1. $(u+v)+w = u+(v+w)$ for all $u, v, w \in V$
2. There is a zero vector $\underline{\underline{0}} \in V$ so that

$$0+v = v+0 = v$$

for all $v \in V$.

3. For all $v \in V$, there is a $-v \in V$ so that

$$v+(-v) = (-v)+v = \underline{\underline{0}}$$

(V, +) is abelian

4. $u+v = v+u$ for all $u, v \in V$

(V, +, ·) is an $\mathbb{R}$-module

5. $c(u+v) = cu+cv$ for all $c \in \mathbb{R}$, all $u, v \in V$
6. $(c+d)u = cu+du$ for all $c, d \in \mathbb{R}$, all $u \in V$
7. $(cd)v = c(dv)$ for all $c, d \in \mathbb{R}$, $v \in V$

↑ is

unitary 8. $1 \cdot v = v$ for all $v \in V$

Remarks: 1. 0 and $-v$ are unique.

For instance, if 0 and $0'$ satisfy axiom 2. Then

$$0 = 0 + 0' = 0'.$$

2. We may replace $\mathbb{R}$ w/ an arbitrary field (e.g. $\mathbb{Q}$, $\mathbb{C}$) and most of our results will still be true.

3. For any $v \in V$ and $c \in \mathbb{R}$:

$$\underset{\text{scalar}}{0}\, v = \underset{\text{vector}}{0}$$

$$\left(\cancel{0v} = (0+0)v = 0v + \cancel{0v} \right)$$

$$c\,0 = 0$$

$$(-1)v = -v$$

$$\left(\begin{aligned} &(-1)v + v = (-1)v + 1 \cdot v = (-1+1)v = 0v = 0 \\ &\Rightarrow (-1)v = -v \checkmark \end{aligned} \right)$$

Examples:

1. $\mathbb{R}^n$ is a vector space for any $n \in \mathbb{N}$.

2. $M_{m\times n}(\mathbb{R}) = \{\text{real } m\times n \text{ matrices}\}$
   is a vector space w/ ordinary matrix addition + scalar mult.

3. $V = \{\text{all sequences of real \#'s}\}$

$$\{x_1, x_2, x_3, x_4, \ldots\} \text{ w/ } x_i \in \mathbb{R}$$

If x, $y = \{y_1, y_2, y_3, \ldots\}$, we define:

$$x + y = \{x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots\}$$

$$cx = \{cx_1, cx_2, cx_3, \ldots\}$$

Then V w/ these operations is a vector space, with

$$0 = \{0, 0, 0, \ldots\}$$

$$-x = \{-x_1, -x_2, -x_3, \ldots\}$$

4. $\mathbb{R}[X] = \{$ all polys w/ real coeffs $\}$

$$a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n = p(X)$$

w/ $a_i \in \mathbb{R}$, $n \geq 0$

We will often write

$$p(X) = \sum_i a_i X^i \quad \text{w/ } i \geq 0,$$

$a_i = 0$ for $i >> 1$

Given $q(X) = \sum_i b_i X^i$ we define

$$(p+q)(X) = \sum_i (a_i + b_i) X^i$$

$$(cp)(X) = \sum_i (ca_i) X^i$$

$\mathbb{R}[X]$ w/ these operations is a vector space. We have

$$0 = 0 + 0 \cdot X + 0 \cdot X^2 + \cdots$$

$$-p(X) = \sum_i (-a_i) X^i$$

5. $V = \{ f : D \to \mathbb{R} \}$

D ← some set (domain)

Given $f, g \in V$ we define

$$(f+g)(x) = f(x) + g(x) \text{ for all } x \in D$$

$$(cf)(x) = c \cdot f(x) \text{ for all } x \in D$$

V w/ these ops. is a v.s.:

$$0(x) = 0 \text{ for all } x \in D$$

$$(-f)(x) = -(f(x)) \text{ for all } x \in D$$