Bases, Dimension + Coordinates

Recall: V a v.s., $\mathcal{B} \subseteq V$ is a **basis** for V iff:

1. $V = \operatorname{Span} \mathcal{B}$
2. $\mathcal{B}$ is lin. ind.

Ex: $M_{m \times n}(\mathbb{R}) = \{$all real $m \times n$ matrices$\}$

Want: Basis for $M_{m \times n}(\mathbb{R})$.

Consider:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a_{21} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}$$

$$= a_{11}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_{12}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$+ a_{21}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_{22}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

This shows

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis for $M_2(\mathbb{R})$.

<u>In general</u>:

$$\underline{\underline{E_{ij}}} = \begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & 1 & & \vdots \\ 0 & \cdots & & 0 \end{pmatrix} \in M_{m\times n}$$

ij position

$$\mathcal{B} = \left\{ E_{ij} \;\middle|\; \underline{\underline{1 \le i \le m}},\ \underline{\underline{1 \le j \le n}} \right\}$$

is a basis for $M_{m \times n}(\mathbb{R})$.

<u>Note</u>: $|\mathcal{B}| = mn$.

<u>Theorem</u>: <u>Every</u> v.s. has a basis.
In fact, any lin. ind. set in V
is contained in a basis for V.

<u>Remark</u>: We proved this directly
for f.g.v.s. The general result
requires <u>Zorn's Lemma</u> or
<u>Transfinite Induction</u>.

We also proved:

<u>Theorem</u>: Let V be a f.g.v.s.
Then every spanning set of V

contains a basis.

Remark: I'm not sure if this is still true for non-f.g. v.s. ...

Coordinates Let V be a f.g. v.s. w/ basis $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$.

Define

$$T : \mathbb{R}^n \longrightarrow V$$

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \longmapsto \sum_i a_i v_i$$

Then:

1. T preserves vector add'n and scalar mult.:

$$T\left(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}\right) = T\begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

$$= \sum_i (a_i + b_i) v_i = \sum_i (a_i v_i + b_i v_i)$$

$$= \sum_i a_i v_i + \sum_i b_i v_i$$

$$= T\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + T\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \checkmark$$

2. T is one-to-one and onto.

<u>Onto</u>: $V = \operatorname{Span} \mathcal{B}$ $\checkmark$

<u>1-1</u>: Since T is a lin. trans., suffices to show $\ker T = \{0\}$.

This is true since $\mathcal{B}$ is lin. ind.!

Properties 1 & 2 say that T is an <u>isomorphism</u>, and we write

$$\mathbb{R}^n \cong V.$$

By #2 above, **every** $v \in V$ can be written as

$$v = \sum_i a_i v_i$$

for **unique** $a_1, a_2, \ldots, a_n$. We define the **$\mathcal{B}$-coords.** of v to be:

$$[v]_{\mathcal{B}} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = T^{-1}(v).$$

Ex: $\mathcal{B} = \{1, X, X^2, \ldots, X^n\}$ is a basis for $\mathbb{P}_n$. Given

$$f(X) = a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n$$

we have:

$$[f]_{\mathcal{B}} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} \text{ and } \mathbb{P}_n \cong \mathbb{R}^{n+1}$$

Because T is an iso., facts about lin. ind./spanning can be "translated" between V and $\mathbb{R}^n$.

For instance:

$$\underset{\text{lin. ind.}}{S \subseteq V} \iff \underset{\text{lin. ind.}}{[S]_{\mathcal{B}} \subseteq \mathbb{R}^n}$$

$$\underset{\text{lin. ind.}}{S \subseteq \mathbb{R}^n} \implies |S| \leq n$$

$$\Big\downarrow T$$

$$\underset{\text{lin. ind.}}{S \subseteq V} \implies |S| \leq n = |\mathcal{B}|$$

As before, this proves:

<u>Theorem</u>: Let V be a f.g.v.s. w/ basis $\mathcal{B}$. If $S \subseteq V$ is lin. ind., then $|S| \leq |\mathcal{B}|$. In particular, <u>every</u> basis of V has size $|\mathcal{B}|$.

<u>Def</u>: If V is a f.g.v.s. w/ basis $\mathcal{B}$, we define the <u>dimension</u> of V to be

$$\dim V = |\mathcal{B}|.$$

Work above shows $\dim V$ is well-defined.

<u>Remark</u>: 1. In fact, the notion of dimension is well-defined in <u>any</u> v.s. (i.e. all bases have the same <u>cardinality</u>), but

This is too hard for us.

2. Now we ~~see~~ that

$$\text{f.g.v.s.} = \text{f.d.v.s.}$$

<u>Theorem</u>: If $\dim V = n$, then $V \cong \mathbb{R}^n$ (by coord. map T).

<u>Ex</u>: 1. Since $\mathcal{B} = \{1, x, x^2, \ldots, x^n\}$ is a basis for $\mathbb{P}_n$, we have

$$\dim \mathbb{P}_n = n+1$$

and

$$\mathbb{P}_n \cong \mathbb{R}^{n+1}$$

2. Since $\mathcal{B} = \{ E_{ij} \mid 1 \le i \le m, 1 \le j \le n \}$ is a basis for $M_{m \times n}(\mathbb{R})$, so

$$\dim M_{m\times n}(\mathbb{R}) = mn$$

and

$$M_{m\times n}(\mathbb{R}) \cong \mathbb{R}^{mn}$$