Exam 2 Corrections

- You may <u>redo</u> any exam problem that you lost points on.
- Redos will be graded out of <u>7 points</u> and these points will <u>replace</u> your original scores.
  (So no need to redo a problem for which you already have ≥ 7 pts.)
- Submit corrections through <u>Gradescope</u> by <u>4/21</u>. Please include a cover sheet indicating

which problems you are resubmitting.

$V = \{\text{all sequences in } \mathbb{R}\}$

$H = \{\text{eventually zero sequences}\}$

Want: Basis for H.

Motivation: Standard basis for $\mathbb{R}^n$:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ \dots,\ e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

This suggests: Define

$$e_i = \{0, 0, 0, \dots, \underset{\uparrow\ i^{th}\text{ term}}{1}, 0, 0, \dots\}$$

Let $\{a_n\} \in V$. Then:

$$\boxed{\{a_1, a_2, a_3, \ldots\}} = \sum_i a_i e_i$$

$$= \{a_1, 0, 0, \ldots\} = a_1 \underbrace{\{1, 0, 0, \ldots\}}_{e_1}$$

$$+$$

$$\{0, a_2, 0, \ldots\} = a_2 \underbrace{\{0, 1, 0, \ldots\}}_{e_2}$$

$$+$$

$$\{0, 0, a_3, 0, \ldots\} = a_3 \underbrace{\{0, 0, 1, 0, \ldots\}}_{e_3}$$

$$+$$

$$\vdots \qquad \vdots$$

$$\overset{?}{\Longrightarrow} \quad \cancel{V} = \boxed{\text{Span}\,\underbrace{\{e_1, e_2, e_3, \ldots\}}_{\text{are lin. ind.}} = H}$$

Linear Transformations

Let V, W be v.s. A function

$$T : V \longrightarrow W$$

is a <u>linear transformation</u> iff:

$$\boxed{\begin{array}{l} 1.\ T(x+y) = T(x)+T(y) \\ \quad \text{for all } x,y \in V \\ 2.\ T(cx) = c \cdot T(x) \text{ for} \\ \quad \text{all } c \in \mathbb{R},\ x \in V. \end{array}}$$

As before, one can show that if T is linear, then:

$$T(0) = 0$$

$$T(-v) = -T(v)$$

$$T\Big(\sum_i a_i v_i\Big) = \sum_i a_i T(v_i)$$

T respects/preserves linear combos.

Ex: 1. Consider $\frac{d}{dx} : C^1(\mathbb{R}) \to C^0(\mathbb{R})$

$$f(x) \longmapsto \frac{d}{dx}(f(x))$$

In Calculus you learn:

So $\frac{d}{dx}$ is a lin. trans.! $\begin{cases} \frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx} \\ \frac{d}{dx}(c \cdot f) = c\frac{df}{dx} \end{cases}$

We can also use $\frac{d}{dx}$ as a lin. trans. on <u>other</u> spaces:

$$\frac{d}{dx} : \mathbb{P} \longrightarrow \mathbb{P}$$

$$\frac{d}{dx} : \mathbb{P}_n \longrightarrow \mathbb{P}_{n-1}$$

2. Because

$$\frac{d^k}{dx^k} = \underbrace{\frac{d}{dx} \circ \frac{d}{dx} \circ \cdots \circ \frac{d}{dx}}_{k \text{ times}}$$

and compositions of lin. trans. are linear (exercise), we find that d^k/dx^k is a lin. trans., e.g.

$$\frac{d^k}{dx^k} : C^k(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$$

$$\frac{d^k}{dx^k} : \mathbb{P} \longrightarrow \mathbb{P}$$

$$\frac{d^k}{dx^k} : \mathbb{P}_n \longrightarrow \mathbb{P}_{n-k}$$

3. Even more generally, given $a_0, a_1, \dots, a_n \in \mathbb{R}$, the <u>differential operator</u>

$$D = a_0 + a_1 \frac{d}{dx} + a_2 \frac{d^2}{dx^2} + \dots + a_n \frac{d^n}{dx^n}$$

is a lin. trans. (since lin. combos. of lin. trans. are lin.).

4. Fix $A \in M_n(\mathbb{R})$, define

$$T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$$
$$B \longmapsto AB - BA.$$

This is a lin. trans.:

- $T(B+C) = A(B+C) - (B+C)A$
$$= AB + AC - BA - CA$$
$$= (AB - BA) + (AC - CA)$$
$$= T(B) + T(C) \checkmark$$

- $T(cB) = A(cB) - (cB)A$
$$= c(AB) - c(BA)$$
$$= c(AB - BA) = cT(B) \checkmark$$

5. Suppose V is a v.s. of functions on $\mathbb{R}$ (e.g. $C^1(\mathbb{R})$, $C^0(\mathbb{R})$, $\mathbb{P}$, ...)

Let $a \in \mathbb{R}$. The <u>evaluation map</u>

$$\begin{aligned} E_a : V &\longrightarrow \mathbb{R} \\ f &\longmapsto f(a) \end{aligned}$$

is a linear trans.

6. Let $V = \mathbb{R}^{\mathbb{N}}$

$= \{\text{all sequences in } \mathbb{R}\}$

Define:

$$L : \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}^{\mathbb{N}} \quad \left(\begin{array}{c}\underline{\text{Left}} \\ \underline{\text{shift}}\end{array}\right)$$

$$\{a_1, a_2, a_3, \dots\} \longmapsto \{a_2, a_3, a_4, \dots\}$$

$$R : \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}^{\mathbb{N}} \quad \left(\begin{array}{c}\underline{\text{Right}} \\ \underline{\text{shift}}\end{array}\right)$$

$$\{a_1, a_2, a_3, \dots\} \longmapsto \{0, a_1, a_2, a_3, \dots\}$$

Then L, R are lin. trans.

Let $T : V \longrightarrow W$ be a lin. trans. of v.s.. Define:

Subspace of V ← $\ker T = \{ v \in V \mid T(v) = 0 \}$

Subspace of W ← $\operatorname{im} T = \{ T(v) \mid v \in V \}$

We say:

T is <u>one-to-one</u> iff: → <u>injective</u>

$$T(x) = T(y) \Rightarrow x = y$$

T is <u>onto</u> iff:
↳ <u>surjective</u>

For all $w \in W$, there is a $v \in V$ so that $T(v) = w$.

$$\Updownarrow$$

$$\operatorname{im} T = W$$

Theorem: If $T: V \to W$ is a lin. trans. of v.s., then:

$$T \text{ is } 1\text{-}1 \iff \ker T = \{0\}$$

Proof:

($\Rightarrow$) Suppose T is 1-1. If $v \in \ker T$, we have:

$$T(v) = 0 = T(0)$$

$\Rightarrow v = 0$

$$\Rightarrow \ker T = \{0\}.$$

($\Leftarrow$) Suppose $\ker T = \{0\}$, and that

$$T(x) = T(y). \Rightarrow T(x) - T(y) = 0$$

$$\Rightarrow T(x-y) = 0$$

$$\Rightarrow x - y \in \ker T = \{0\}$$

$$\Rightarrow x - y = 0$$

$$\Rightarrow x = y. \quad \blacksquare$$

Ex: What are kernels/images of the prev. examples? Which are 1-1/onto?

1. $\frac{d}{dx}: C^1(\mathbb{R}) \to C^0(\mathbb{R})$

$$f \in \ker \frac{d}{dx} \iff \frac{df}{dx} = 0$$
$$\iff f = \text{const.}$$

$$\boxed{\ker \frac{d}{dx} = \{\text{const. fns.}\}} \Rightarrow d/dx \text{ is } \underline{\text{not}} \text{ 1-1.}$$

$$\boxed{\operatorname{im} \frac{d}{dx} = C^0(\mathbb{R})} \text{ by FTOC}$$

$$\Rightarrow d/dx \ \underline{\text{is}} \text{ onto.}$$

2. $\frac{d^k}{dx^k}: C^k(\mathbb{R}) \to C^0(\mathbb{R})$

$$f \in \ker \frac{d^k}{dx^k} \iff \frac{d^k f}{dx^k} = 0$$
$$\iff f \in \boxed{\mathbb{P}_{k-1} = \ker \frac{d^k}{dx^k}}$$

$\Rightarrow \dfrac{d^k}{dx^k}$ is <u>not</u> 1-1

$\dfrac{d^k}{dx^k}$ is onto by FTOC

3. $D = a_0 + a_1 \dfrac{d}{dx} + a_2 \dfrac{d^2}{dx^2} + \cdots + a_n \dfrac{d^n}{dx^n}$

$f \in \ker D \iff f$ solves the linear diff. eq.

$$a_0 f + a_1 \frac{df}{dx} + a_2 \frac{d^2 f}{dx^2} + \cdots + a_n \frac{d^n f}{dx^n} = 0$$

(So solns of homog. linear ODEs are elements in the kernel of a differential operator)

4. $T(B) = AB - BA$

$\ker T = \{ B \in M_n(\mathbb{R}) \mid AB - BA = 0 \}$

$AB = BA$