Computing w/ Coordinates

Given an ordered basis

$$\mathcal{B} = \{v_1, v_2, \dots, v_n\}$$

for a f.d.v.s. V (so $\dim V = n$), we have the $\mathcal{B}$-coordinate map

$$T_{\mathcal{B}} : \mathbb{R}^n \longrightarrow V$$

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \longmapsto a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

This is a 1-1, onto, linear transformation, i.e. an <u>isomorphism</u> between $\mathbb{R}^n$ and V.

The inverse map $T_{\mathcal{B}}^{-1} = [\cdot]_{\mathcal{B}}$ transforms linear algebra in V to lin. alg. in $\mathbb{R}^n$, where we already have tools.

Let V, W be f.d.v.s. w/ bases $\mathcal{B}$ and $\mathcal{C}$ (resp.), and let

$$T : V \longrightarrow W$$

be a lin. trans. If $v \in V$, let

$$[v]_{\mathcal{B}} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

i.e. $v = \sum_i a_i v_i$. $\longleftarrow$

$$\Rightarrow T(v) = T\left(\sum_i a_i v_i\right)$$

$$= T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n)$$

Linearity of T

$$= a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n)$$

Since $[\cdot]_{\mathcal{C}}$ is a lin. trans. $W \to \mathbb{R}^m$ ($m = |\mathcal{C}| = \dim W$), we may apply it above to get

$$[T(v)]_{\mathcal{C}} = a_1 [T(v_1)]_{\mathcal{C}} + a_2 [T(v_2)]_{\mathcal{C}} + \cdots + a_n [T(v_n)]_{\mathcal{C}}$$

$$= \underbrace{\begin{pmatrix} [T(v_1)]_{\mathcal{C}} & [T(v_2)]_{\mathcal{C}} & \cdots & [T(v_n)]_{\mathcal{C}} \end{pmatrix}}_{[T]^{\mathcal{C}}_{\mathcal{B}}} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

Matrix of T relative to $\mathcal{B}, \mathcal{C}$.

$$= [T]^{\mathcal{C}}_{\mathcal{B}} \cdot [v]_{\mathcal{B}}$$

$$\boxed{[T(v)]_{\mathcal{C}} = [T]^{\mathcal{C}}_{\mathcal{B}} [v]_{\mathcal{B}}}$$

Remarks:

1. This shows that at the level of coordinates, T is given by matrix multiplication.

2. This whole set-up can be described by the commutative diagram:

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "{[\cdot]_{\mathcal{B}}}"', "\wr"] & W \arrow[d, "\wr"', "{[\cdot]_{\mathcal{C}}}"] \\
\mathbb{R}^n \arrow[r, "{[T]_{\mathcal{B}}^{\mathcal{C}}}"] & \mathbb{R}^m
\end{tikzcd}$$

3. If $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, $\mathcal{E}$ is the standard basis, and $T: \mathbb{R}^n \to \mathbb{R}^m$, then

$$[T] = [T]_{\mathcal{E}}^{\mathcal{E}}.$$

Ex:

1. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Define

$$T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$$
$$B \longmapsto AB - BA,$$

which we know is a lin. trans.

As our bases we take

$$\mathcal{B} = \mathcal{C} = \boxed{\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}}$$

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2 \\ 3 & 0 \end{pmatrix} \xrightarrow{[\cdot]_{\mathcal{C}}} \begin{pmatrix} 0 \\ -2 \\ 3 \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -3 \\ 0 & 3 \end{pmatrix} \xrightarrow{[\cdot]_{\mathcal{E}}} \begin{pmatrix} -3 \\ -3 \\ 0 \\ 3 \end{pmatrix}$$

$$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \underline{\qquad\qquad\qquad\qquad}$$

$$= \begin{pmatrix} 2 & 0 \\ 3 & -2 \end{pmatrix} \xrightarrow{[\cdot]_{\mathcal{E}}} \begin{pmatrix} 2 \\ 0 \\ 3 \\ -2 \end{pmatrix}$$

$$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \underline{\qquad\qquad\qquad\qquad}$$

$$= \begin{pmatrix} 0 & 2 \\ -3 & 0 \end{pmatrix} \xrightarrow{[\cdot]_{\mathcal{E}}} \begin{pmatrix} 0 \\ 2 \\ -3 \\ 0 \end{pmatrix}$$

So the matrix of T rel. $\mathcal{B}, \mathcal{E}$ is:

$$\left(\left[T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_{\mathcal{E}} \left[T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]_{\mathcal{E}} \left[T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]_{\mathcal{E}} \left[T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]_{\mathcal{E}} \right)$$

$$= \begin{pmatrix} 0 & -3 & 2 & 0 \\ -2 & -3 & 0 & 2 \\ 3 & 0 & 3 & -3 \\ 0 & 3 & -2 & 0 \end{pmatrix} = [T]^{\mathcal{C}}_{\mathcal{B}}$$

How can we use this?

$$B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$$

Suppose we want $T(B)$. We have two options:

1. Compute $T(B) = AB - BA$ directly... this is annoying.

2. Multiply $[B]_{\mathcal{B}}$ by $[T]^{\mathcal{C}}_{\mathcal{B}}$, and apply inverse $\mathcal{C}$-coord. map.

$$\begin{pmatrix} 0 & -3 & 2 & 0 \\ -2 & -3 & 0 & 2 \\ 3 & 0 & 3 & -3 \\ 0 & 3 & -2 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -15 \\ 15 \\ 5 \end{pmatrix} \xrightarrow{[\cdot]_{\mathcal{C}}^{-1}} \begin{pmatrix} -5 & -15 \\ 15 & 5 \end{pmatrix}$$

$[T]^{\mathcal{C}}_{\mathcal{B}}$ $\quad [B]_{\mathcal{B}}$ $\quad [T(B)]_{\mathcal{C}}$

That is:

$$T\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}$$

$$= \begin{pmatrix} -5 & -15 \\ 15 & 5 \end{pmatrix}$$

To compute $\ker T = \{ B \mid AB = BA \}$, we compute $\ker [T]^{\mathcal{C}}_{\mathcal{B}}$ and convert via $[\cdot]^{-1}_{\mathcal{B}}$.

$$[T]^{\mathcal{C}}_{\mathcal{B}} = \begin{pmatrix} 0 & -3 & 2 & 0 \\ -2 & -3 & 0 & 2 \\ 3 & 0 & 3 & -3 \\ 0 & 3 & -2 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So $[T]^{\mathcal{C}}_{\mathcal{B}} \vec{x} = \vec{0}$ iff

$\vec{x} \to \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

$$x_1 + x_3 - x_4 = 0$$
$$x_2 - \frac{2}{3} x_3 = 0$$

x_3, x_4 free

$$\iff \begin{aligned} x_1 &= -x_3 + x_4 \\ x_2 &= \frac{2}{3} x_3 \end{aligned}$$

x_3, x_4 free

$$\iff \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_3 + x_4 \\ 2/3 \cdot x_3 \\ x_3 \\ x_4 \end{pmatrix}$$

$$= x_3 \begin{pmatrix} -1 \\ 2/3 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

So $\left\{ \begin{pmatrix} -3 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis

for $\mathrm{Null}\, [T]^{e}_{\mathcal{B}}$.

$$\Big\downarrow [\cdot]^{-1}_{\mathcal{B}}$$

$\left\{ \begin{pmatrix} -3 & 2 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a

basis for $\ker T$. So

$$\boxed{\ker T = \{ B \mid AB = BA \} = \mathrm{Span} \left\{ \begin{pmatrix} -3 & 2 \\ 3 & 0 \end{pmatrix}, I \right\}}$$

Remark: One can show $\{A, I\}$ is also a basis. This means:

$$\{B \mid AB = BA\} = \{aA + bI \mid a, b \in \mathbb{R}\}$$

Is T 1-1? No: $\dim \ker T = 2 \neq 0$.

Is T onto? Work above show

$$\begin{aligned} \dim \operatorname{im} T &= \dim \operatorname{im} [T]_{\mathcal{B}}^{e} \\ &= \# \text{ pivot cols.} \\ &= 2 \neq 4 = \dim M_2(\mathbb{R}) \end{aligned}$$

$\Rightarrow T$ is not onto.

In fact

$$\operatorname{im} T = \operatorname{Span}\left\{ \begin{pmatrix} 0 & -2 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} -3 & -3 \\ 0 & 3 \end{pmatrix} \right\}$$

↳ basis for im T