# More Examples Using Coordinates

**1.** Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$ (distinct) and define

$$T : \mathbb{P}_{n-1} \to \mathbb{R}^n$$
$$f(X) \mapsto \begin{pmatrix} f(a_1) \\ \vdots \\ f(a_n) \end{pmatrix}$$

Let $\mathcal{B} = \{1, X, \ldots, X^{n-1}\}$ (a basis for $\mathbb{P}_{n-1}$), $\mathcal{E}$ = std. basis of $\mathbb{R}^n$.

The matrix for $T$ rel. these bases is:

$$\begin{pmatrix} T(1) & T(X) & T(X^2) & \cdots & T(X^{n-1}) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & & a_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix} = V(a_1, \ldots, a_n)$$

Claim: $T$ is 1-1

$f(X) \in \ker T \iff f(a_1) = f(a_2) = \cdots = f(a_n) = 0$

$\Rightarrow$ $f$ (poly. deg $\leq n-1$) has $a_1, a_2, \ldots, a_n$ (all dist.) as roots.

$n$ roots

$\Rightarrow f(X) \equiv 0$

$\Rightarrow \ker T = \{0\}$

$\Rightarrow T$ is 1-1

Consequences:

1. $V(a_1, \ldots, a_n)$ is invertible!

(by inv. matr. thm.)

2. $V(a_1, \ldots, a_n)$ represents an onto trans. $\Rightarrow$ $T$ is onto!

3. Given <u>any</u> $b_1, \dots, b_n \in \mathbb{R}$,

$\exists!\ f(x) \in \mathbb{P}_{n-1}$ w/

$$f(a_i) = b_i \quad \forall i.$$

<u>2.</u> Let $V_n = e^x \cdot \mathbb{P}_n = \{ e^x f(x) \mid f(x) \in \mathbb{P}_n \}$

Can show $V_n \leq C'(\mathbb{R})$.

Let $f(x) \in \mathbb{P}_n$ and notice that

$$\frac{d}{dx}\left(e^x f(x)\right) = e^x f'(x) + e^x f(x)$$
$$= e^x \big( \underbrace{f'(x) + f(x)}_{\text{poly. of degree } = \deg f \leq n} \big) \in V_n$$

So we have an <u>endomorphism</u>

$$\frac{d}{dx} : V_n \longrightarrow V_n.$$

Easy to show that

$$\mathcal{B} = \{ e^x, e^x x, e^x x^2, \dots, e^x x^n \}$$

is a basis for $V_n$. Let's compute

$$\left[ \frac{d}{dx} \right]_{\mathcal{B}}^{\mathcal{B}}$$

$$= \left( \left[ \frac{d}{dx} (e^x) \right]_{\mathcal{B}} \left[ \frac{d}{dx} (e^x x) \right]_{\mathcal{B}} \left[ \frac{d}{dx} (e^x x^2) \right]_{\mathcal{B}} \cdots \left[ \frac{d}{dx} (e^x x^n) \right] \right)$$

$$= \left( [e^x]_{\mathcal{B}} \; [e^x(1+x)]_{\mathcal{B}} \; [e^x(2x+x^2)]_{\mathcal{B}} \cdots \right.$$

$e^x(3x^2+x^3)$

$$\left. \cdots [e^x(nx^{n-1} + x^n)]_{\mathcal{B}} \right)$$

$e^x + e^x x$

$\mathcal{B} = \{ e^x, e^x x, e^x x^2, \dots, e^x x^n \}$

$$= \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 2 & 0 & \cdots & \vdots \\ \vdots & 0 & 1 & 3 & \ddots & \vdots \\ \vdots & \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & 0 & \ddots & n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

<u>Note</u>: $\det = 1 \Rightarrow \left[\frac{d}{dx}\right]_{\mathcal{B}}^{\mathcal{B}}$ is invertible!

$\Rightarrow \frac{d}{dx} : V_n \longrightarrow V_n$ is invertible (1-1 <u>and</u> onto)

$\Rightarrow \int \sim dx : V_n \to V_n$ is <u>also</u> an (inv.) lin. trans.

<u>Moral</u>: To integrate fns. in $V_n$, we multiply their coord. vectors

by $\left(\left[\frac{d}{dx}\right]_{\mathcal{B}}^{\mathcal{B}}\right)^{-1}$.

We can easily invert $\left[\frac{d}{dx}\right]_{\mathcal{B}}^{\mathcal{B}}$ as follows:

$$= I + \underbrace{\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & n \\ & & & 0 \end{pmatrix}}_{N} = I + N$$

$N$ is nilpotent, i.e. $N^{n+1} = 0$

In HW we saw:

$$(I + N)^{-1} = (I - (-N))^{-1}$$

$$= I - N + N^2 - N^3 + N^4 - \cdots (-1)^n N^n$$

Ex: $n=3$. Then

$$N = \begin{pmatrix} 0 & 1 & & \\ & 0 & 2 & \\ & & 0 & 3 \\ & & & 0 \end{pmatrix}$$

$$N^2 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$N^3 = \begin{pmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$N^4 = 0$$

So the inverse of $\left[\frac{d}{dx}\right]_{\mathcal{B}}^{\mathcal{B}}$ is:

$$\begin{pmatrix} 1 & -1 & 2 & -6 \\ & 1 & -2 & 6 \\ & & 1 & -3 \\ & & & 1 \end{pmatrix}$$

This implements integration on $V_3$ (using $\mathcal{B}$-coords.)