Rank

Let $T : V \longrightarrow W$ be a lin. trans. of v.s.

Then the kernel of T is:

$$\ker T = \{ v \in V \mid \boxed{T(v) = 0} \}$$
$$= T^{-1}(\{0\})$$

Theorem: $\ker T \leq V$ and T is 1-1 iff $\ker T = \{0\}$.

Proof: We have:

1. $0 \in \ker T$: $T(0) = 0$ since T is linear ✓

2. $u, v \in \ker T \Rightarrow u + v \in \ker T$:
$T(u+v) = T(u) + T(v) = 0 + 0 = 0$ ✓

3. $u \in \ker T$, $c \in \mathbb{R} \Rightarrow cu \in \ker T$:

So $\ker T$ is a subspace

$$T(cu) = cT(u) = c \cdot 0$$
$$= 0 \checkmark$$

$x, y \in V$:

$$T(x) = T(y) \iff T(x) - T(y) = 0$$
$$\iff T(x-y) = 0$$
$$\iff x - y \in \ker T$$

This shows:

$\ker T = \{0\} \Rightarrow T$ is 1-1

$\ker T \neq \{0\} \Rightarrow T$ is not 1-1

$\blacksquare$

Let $\mathcal{B}$ be a basis for $\ker T$.

Complete $\mathcal{B}$ to a basis for V:

$$\mathcal{B}' = \mathcal{B} \cup \mathcal{C}$$

vectors needed to complete

Claim: $T(\mathcal{C}) = \{T(v) \mid v \in \mathcal{C}\}$ is a basis for $\operatorname{im} T$.

Why? Let $w \in \operatorname{im} T \leq W$.

Then $w = T(v)$ for some $v \in V$.

So

$$w = T\Big(\sum_{u \in \mathcal{B}'} \overset{\text{scalar}}{a(u)} \cdot u\Big)$$

$$= T\Big(\sum_{u \in \mathcal{B}} a(u) \cdot u + \sum_{u \in \mathcal{C}} a(u) \cdot u\Big)$$

$$= T\Big(\underbrace{\cancelto{0}{\sum_{u \in \mathcal{B}} a(u) \cdot u}}_{\text{in } \ker T}\Big) + T\Big(\sum_{u \in \mathcal{C}} a(u) \cdot u\Big)$$

$$= \sum_{u \in \mathcal{C}} a(u) \cdot T(u) \in \operatorname{Span} T(\mathcal{C})$$

So $\operatorname{im} T = \operatorname{Span} T(\mathcal{C})$. ✓

Also, if

$$\sum_{u \in \mathcal{C}} a(u) \cdot T(u) = 0$$

$$\Downarrow$$

$$T\left(\sum_{u \in \mathcal{C}} a(u) \cdot u\right) = 0$$

$$\Rightarrow \sum_{u \in \mathcal{C}} a(u) \cdot u \in \ker T = \operatorname{Span} \mathcal{B}$$

$$\Rightarrow \sum_{u \in \mathcal{C}} a(u) \cdot u = \sum_{u \in \mathcal{B}} a(u) \cdot u$$

$$\Rightarrow \sum_{u \in \mathcal{B}} (-a(u)) \cdot u + \sum_{u \in \mathcal{C}} a(u) \cdot u = 0$$

lin. combo. of vectors

in $\mathcal{B}' = \mathcal{B} \cup \mathcal{C}$ $\leftarrow$ lin. ind.!

$\Rightarrow$ $a(u) = 0$ for every $u \in \mathcal{C}$

So $T(\mathcal{C})$ is lin. ind. ✓

$\blacksquare$

Therefore:

$$\operatorname{rank} T = \dim \operatorname{im} T = |T(\mathcal{C})|$$
$$= |\mathcal{C}|$$
$$\dim V = |\mathcal{B}'| = |\mathcal{B}| + |\mathcal{C}|$$
$$= \dim \ker T + \operatorname{rank} T$$

<u>Rank Theorem</u>: If $T : V \to W$ is a lin. trans. of v.s., then

$$\boxed{\dim V = \operatorname{rank} T + \dim \ker T}$$

If $\mathcal{B}, \mathcal{C}$ are finite bases for V, W (resp.), we have:

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "{[\cdot]_{\mathcal{B}}}"', "\wr"] & W \arrow[d, "{[\cdot]_{\mathcal{C}}}", "\wr"'] \\
\mathbb{R}^n \arrow[r, "{[T]_{\mathcal{B}}^{\mathcal{C}}}"] & \mathbb{R}^m
\end{tikzcd}$$

$n = \dim V \qquad m = \dim W$

Not hard to show:

$$\ker T \longleftrightarrow \operatorname{Null} [T]_{\mathcal{B}}^{\mathcal{C}}$$
$$\operatorname{im} T \longleftrightarrow \operatorname{Col} [T]_{\mathcal{B}}^{\mathcal{C}}$$
$$\operatorname{rank} T \longleftrightarrow \operatorname{rank} [T]_{\mathcal{B}}^{\mathcal{C}}$$

via coords.

Ex: Let $D : \mathbb{P}_n \longrightarrow \mathbb{P}_n$ be given by

$$D(y) = y'' - y' + y.$$

Then D is linear. Let's compute $\ker T$, $\operatorname{im} T$, $\operatorname{rank} T$. Set

$$\mathcal{B} = \{1, X, X^2, \dots, X^n\}.$$

$$T(1) = 1'' - 1' + 1 = 1$$

$$T(X) = 0 - 1 + X = -1 + X$$

$$T(X^2) = 2 - 2X + X^2$$

$$T(X^3) = 6X - 3X^2 + X^3$$

$$T(X^i) = i(i-1)X^{i-2} - iX^{i-1} + X^i$$

$$\vdots$$

$$T(X^n) = n(n-1)X^{n-2} - nX^{n-1} + X^n$$

$$\Big\downarrow [\,\cdot\,]_{\mathcal{B}}$$

$$[D]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 1 & -1 & 2 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 6 & & \vdots \\ \vdots & 0 & 1 & -3 & & 0 \\ \vdots & \vdots & 0 & 1 & & n(n-1) \\ & \vdots & \vdots & 0 & \ddots & -n \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$= I - N \Rightarrow (I-N)^{-1} = I + N + N^2 + N^3 + \cdots$

So $\operatorname{rank} D = \operatorname{rank} [D]_{\mathcal{B}}^{\mathcal{B}}$

$$\boxed{\operatorname{rank} D = n+1}$$

If we apply the rank theorem:

$$\dim \mathbb{P}_n = \dim \ker D + \operatorname{rank} D$$

$$\cancel{n+1} = \dim \ker D + \cancel{(n+1)}$$

$$\Rightarrow \dim \ker D = 0$$

$$\Rightarrow \boxed{\ker D = \{0\}}$$

$\boxed{D \text{ is an isomorphism!}}$

"automorphism"

Consider the following Calc. II problem:

Find a particular sol'n to the second order ODE:

$$y'' - y' + y = x^4 - x^3 + 2$$

Let $n = 4$. Then we must solve

$$D(y) = x^4 - x^3 + 2$$

$$\Big\downarrow [\cdot]_{\mathcal{B}} \quad \mathcal{B} = \{1, x, x^2, x^3, x^4\}$$

$$\underbrace{[D]_{\mathcal{B}}^{\mathcal{B}}} \vec{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

apply the inverse!

We have

$$\vec{x} = \begin{pmatrix} 1 & 1 & 0 & -6 & -24 \\ & 1 & 2 & 0 & -24 \\ & & 1 & 3 & 0 \\ & & & 1 & 4 \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -16 \\ -24 \\ -3 \\ 3 \\ 1 \end{pmatrix} = [p]_{\mathcal{B}}$$

So "the" sol'n to the ODE is:

$$\boxed{p(x) = -16 - 24x - 3x^2 + 3x^3 + x^4}$$

<u>Ex</u>: Let $A \in M_n(\mathbb{R})$ and let

$$J_A = \operatorname{Span}\{I, A, A^2, A^3, \ldots\}$$
$$\leq M_n(\mathbb{R})$$

Define $E_A : \mathbb{R}[X] \longrightarrow J_A$ by

$$E_A(p(X)) = p(A)$$

$$a_0 + a_1 X + a_2 X^2 + \cdots \xrightarrow{E_A} a_0 I + a_1 A + a_2 A^2 + \cdots$$

This is linear. According to the rank theorem

$$\underbrace{\dim \mathbb{R}[X]}_{\infty} = \underbrace{\dim \operatorname{im} E_A}_{\leq \dim M_n(\mathbb{R}) = n^2} + \overbrace{\dim \ker E_A}^{\infty}$$

$$\Rightarrow \quad \ker E_A \neq \{0\}$$

$$\Rightarrow \boxed{\exists \ f(X) \neq 0 \text{ w/ } f(A) = O.}$$

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