Change of Basis

Let V be a f.d.v.s. w/ bases $\mathcal{B}, \mathcal{C}$.

The identity map

$$
\begin{aligned}
I : \overset{\mathcal{B}}{V} &\longrightarrow \overset{\mathcal{C}}{V} \\
v &\longmapsto v
\end{aligned}
$$

is a lin. trans. w/

$$[v]_{\mathcal{C}} = [I(v)]_{\mathcal{C}}$$

$$\boxed{[v]_{\mathcal{C}} = [I]^{\mathcal{C}}_{\mathcal{B}} [v]_{\mathcal{B}}}$$

We call $[I]^{\mathcal{C}}_{\mathcal{B}}$ the <u>change of coords.</u> (or <u>change of basis</u>) <u>matrix</u> (from $\mathcal{B}$ <u>to</u> $\mathcal{C}$).

Notice that (if $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$)

$$[I]^{\mathcal{C}}_{\mathcal{B}} = \left([I(b_1)]_{\mathcal{C}} \; [I(b_2)]_{\mathcal{C}} \cdots [I(b_n)]_{\mathcal{C}} \right)$$

$$\boxed{[I]_{\mathcal{B}}^{\mathcal{C}} = \left([b_1]_{\mathcal{C}} \ [b_2]_{\mathcal{C}} \ \cdots \ [b_n]_{\mathcal{C}} \right)}$$

Notice that we have

$$[v]_{\mathcal{C}} = [I]_{\mathcal{B}}^{\mathcal{C}} [v]_{\mathcal{B}}$$

$$[v]_{\mathcal{B}} = [I]_{\mathcal{C}}^{\mathcal{B}} [v]_{\mathcal{C}}$$

$$\Rightarrow \boxed{[I]_{\mathcal{C}}^{\mathcal{B}} = \left([I]_{\mathcal{B}}^{\mathcal{C}} \right)^{-1}}$$

<u>Remarks</u>:

1. If $V = \mathbb{R}^n$, and $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$ is the standard basis, then

$$[I]_{\mathcal{C}}^{\mathcal{E}} = \left([c_1]_{\mathcal{E}} \ [c_2]_{\mathcal{E}} \ \cdots \ [c_n]_{\mathcal{E}} \right)$$

$$\hookrightarrow = \{c_1, c_2, \ldots, c_n\}$$

$$\boxed{[I]_{\mathcal{C}}^{\mathcal{E}} = \left(c_1 \ \ c_2 \ \ \cdots \ \ c_n \right)}$$

$$\Rightarrow \boxed{[I]^{\mathcal{C}}_{\mathcal{E}} = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix}^{-1}}$$

2. Notice that

$$[I]^{\mathcal{C}}_{\mathcal{B}} = [I]^{\mathcal{C}}_{\mathcal{E}}[I]^{\mathcal{E}}_{\mathcal{B}}$$

$$= \left([I]^{\mathcal{E}}_{\mathcal{C}}\right)^{-1}[I]^{\mathcal{E}}_{\mathcal{B}}$$

$$\boxed{[I]^{\mathcal{C}}_{\mathcal{B}} = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix}^{-1}\begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix}}$$

<u>Ex</u>: 1. If $\mathcal{E} = \{e_1, e_2, e_3\}$ and

$$\mathcal{C} = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\},$$

then

$$[I]^{\mathcal{E}}_{\mathcal{C}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow [I]^{\mathcal{C}}_{\mathcal{E}} = \left([I]^{\mathcal{E}}_{\mathcal{C}}\right)^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus:

$$\left[\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right]_{\mathcal{C}} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$$

i.e. $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ✓

2. Let $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ in $\mathbb{R}^4$

and $\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$.

Then

$$[I]_{\mathcal{C}}^{\mathcal{E}} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{pmatrix}$$

$$\Rightarrow [I]_{\mathcal{E}}^{\mathcal{C}} = \left([I]_{\mathcal{C}}^{\mathcal{E}}\right)^{-1}$$

$$= \frac{1}{4}\begin{pmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Thus,

$$\left[\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}\right]_C = \frac{1}{4}\begin{pmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -1 \\ 5/2 \\ -1/2 \end{pmatrix}$$

3. Let $V = \mathbb{P}_n$ w/ $\mathcal{B} = \{1, x, x^2, \dots, x^n\}$.

Set $C = \{1, x-a, (x-a)^2, \dots, (x-a)^n\}$,

where $a \in \mathbb{R}$.

To compute $[I]_C^{\mathcal{B}} = \Big([1]_{\mathcal{B}} [x-a]_{\mathcal{B}} [(x-a)^2]_{\mathcal{B}} \cdots [(x-a)^n]_{\mathcal{B}} \Big)$

we "recall" the **binomial theorem:**

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n} a^0 b^n$$

$$= \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j$$

↳ **Binomial coeffs.** :

$$\binom{n}{j} = \frac{n!}{j!\,(n-j)!}$$

Pascal's Δ:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & & & & \vdots & & & & \end{array}$$

$$(X-a)^k = \sum_{j=0}^{k} \binom{k}{j} (-a)^{k-j} X^j$$

↳ $\mathcal{B}$-coords.

So:

$$[I]^{\mathcal{B}}_{\mathcal{C}} = \begin{pmatrix} 1 & -a & a^2 & -a^3 & \cdots & \vdots \\ 0 & 1 & -2a & 3a^2 & \cdots & \vdots \\ \vdots & 0 & 1 & -3a & \cdots & \vdots \\ \vdots & \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & 0 & & -na \\ 0 & 0 & 0 & 0 & & 1 \end{pmatrix}$$

Notice that:

$$X^k = (X - a + a)^k$$

$$= \sum_{j=0}^{k} \underbrace{\binom{k}{j} a^{k-j}}_{\hookrightarrow\ \mathcal{C}\text{-coords. of } X^k} (X-a)^j$$

$$\Rightarrow [I]^{\mathcal{C}}_{\mathcal{B}} = \begin{pmatrix} 1 & a & a^2 & a^3 & \cdots \\ 0 & 1 & 2a & 3a^2 & \cdots \\ 0 & 0 & 1 & 3a & \ddots \\ \vdots & \vdots & 0 & 1 & \\ \vdots & \vdots & \vdots & 0 & \\ 0 & 0 & 0 & 0 & \end{pmatrix}$$

So $[I]^{\mathcal{C}}_{\mathcal{B}} = \left([I]^{\mathcal{B}}_{\mathcal{C}}\right)^{-1}$ can be "computed" by removing all negative signs from $\mathcal{C}$.

Change of Coords. in Lin. Trans.

Let $T: V \to V$ be a lin. trans., V f.d.v.s. w/ bases $\mathcal{B}$ and $\mathcal{C}$. Set

$$[T]_{\mathcal{B}}^{\mathcal{B}} = [T]_{\mathcal{B}}$$

$$[T]_{\mathcal{C}}^{\mathcal{C}} = [T]_{\mathcal{C}}$$

How related?

Then

$$\boxed{[T]_{\mathcal{C}} \cdot [v]_{\mathcal{C}}} = [T(v)]_{\mathcal{C}}$$

$$= [I]_{\mathcal{B}}^{\mathcal{C}} [T(v)]_{\mathcal{B}}$$

$$= [I]_{\mathcal{B}}^{\mathcal{C}} [T]_{\mathcal{B}} [v]_{\mathcal{B}}$$

$$= \boxed{[I]_{\mathcal{B}}^{\mathcal{C}} [T]_{\mathcal{B}} [I]_{\mathcal{C}}^{\mathcal{B}} \cdot [v]_{\mathcal{C}}}$$

for all $v \in V$.

$$\Rightarrow [T]_{\mathcal{C}} = [I]_{\mathcal{B}}^{\mathcal{C}} [T]_{\mathcal{B}} [I]_{\mathcal{C}}^{\mathcal{B}}$$

$$\boxed{[T]_{\mathcal{C}} = [I]_{\mathcal{B}}^{\mathcal{C}} [T]_{\mathcal{B}} \left([I]_{\mathcal{B}}^{\mathcal{C}}\right)^{-1}}$$

Conjugation by C

Def'n: We say $A, B \in M_n(\mathbb{R})$ are <u>similar</u> if $A = CBC^{-1}$ for some (invertible) $C \in M_n(\mathbb{R})$.

<u>Theorem</u>: Let $A, B \in M_n(\mathbb{R})$. Then

$\Downarrow$ A is similar to B iff A, B represent the <u>same</u> lin. trans. on $\mathbb{R}^n$ (w/ respect to diff. bases). $\Uparrow$