HW 13.1, #3:

$F_1 = F_2 = 1$, $\boxed{F_{n+2} = F_{n+1} + F_n}$ $(n \geq 1)$

$a = b = 1$

By #1b, $\{F_n\} \in \ker T$, where

$T = L^2 - L - I$.

By #2c, $\{F_n\} \in \ker T \iff$

$$\{F_n\} = a_1 \{r_1^{n-1}\} + a_2 \{r_2^{n-1}\}$$

where r_1, r_2 are roots of

$$\boxed{X^2 - X - 1}$$

(provided $r_1 \neq r_2$).

↳ Use quad. formula to find r_1, r_2.

$$a_1 \{r_1^{n-1}\} + a_2 \{r_2^{n-1}\}$$

$$= \{\boxed{a_1 r_1^{n-1} + a_2 r_2^{n-1}}\}$$

$$= \{a_1 + a_2,\ a_1 r_1 + a_2 r_2,\ \ldots\}$$

$$= \{1,\ 1,\ \ldots\}$$

$$a_1 + a_2 = 1$$
$$a_1 r_1 + a_2 r_2 = 1 \quad \rightsquigarrow \begin{pmatrix} 1 & 1 \\ r_1 & r_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Change of Basis / Coordinates

<u>Recall</u>: If $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$, $\mathcal{C} = \{c_1, c_2, \ldots, c_n\}$ are bases for $\mathbb{R}^n$, then

$$[I]_{\mathcal{B}}^{\mathcal{C}} = (c_1\ c_2\ \cdots\ c_n)^{-1}(b_1\ b_2 \cdots b_n)$$

$\underline{\text{Ex}}$: Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\}$,

$\mathcal{C} = \left\{ \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} \right\}$

Find $[I]_{\mathcal{B}}^{\mathcal{C}}$, $[I]_{\mathcal{C}}^{\mathcal{B}}$.

$$[I]_{\mathcal{B}}^{\mathcal{C}} = \underbrace{\begin{pmatrix} -4 & 2 & 4 \\ 3 & -5 & 0 \\ 1 & 1 & -3 \end{pmatrix}^{-1}}_{\mathcal{C}} \underbrace{\begin{pmatrix} 1 & 4 & 0 \\ -4 & -2 & 0 \\ -1 & -5 & 2 \end{pmatrix}}_{\mathcal{B}}$$

In general, if A is invertible and B has the right dimensions, then:

$$\begin{pmatrix} A & B \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} I & A^{-1}B \end{pmatrix}$$

In this case:

$$\begin{pmatrix} -4 & 2 & 4 & 1 & 4 & 0 \\ 3 & -5 & 0 & -4 & -2 & 0 \\ 1 & 1 & -3 & -1 & -5 & 2 \end{pmatrix}$$

$$\xrightarrow{\text{RREF}} \left(\begin{array}{ccc} 1 & & \\ & 1 & \\ & & 1 \end{array} \right. \underbrace{\begin{array}{ccc} 9/2 & 6 & -4 \\ 7/2 & 4 & -12/5 \\ 3 & 5 & -14/5 \end{array}}_{[I]_{\mathcal{B}}^{\mathcal{C}}} \left. \vphantom{\begin{array}{c} 1 \\ 1 \\ 1 \end{array}} \right)$$

$$[I]_{\mathcal{C}}^{\mathcal{B}} = \left([I]_{\mathcal{B}}^{\mathcal{C}} \right)^{-1}$$

$$= \begin{pmatrix} -2/7 & 8/7 & -4/7 \\ -13/14 & 3/14 & 8/7 \\ -55/28 & 45/28 & 15/14 \end{pmatrix}$$

If $T: V \to V$ is a lin. trans. and $\dim V < \infty$, and $\mathcal{B}, \mathcal{C}$ are bases for V:

$$[T]_{\mathcal{B}}^{\mathcal{B}} := [T]_{\mathcal{B}}$$

$$[T]_{\mathcal{C}}^{\mathcal{C}} := [T]_{\mathcal{C}}$$

We showed

$$[T]_{\mathcal{C}} = \underbrace{[I]_{\mathcal{B}}^{\mathcal{C}} [T]_{\mathcal{B}} \boxed{\underbrace{[I]_{\mathcal{C}}^{\mathcal{B}}}_{\left([I]_{\mathcal{B}}^{\mathcal{C}}\right)^{-1}}}}_{\text{Conjugate of } [T]_{\mathcal{B}} \text{ by } [I]_{\mathcal{B}}^{\mathcal{C}}}$$

i.e. $[T]_{\mathcal{C}}$ and $[T]_{\mathcal{B}}$ are <u>similar</u>.

[We say $A, B \in M_n(\mathbb{R})$ are <u>similar</u>

if $\exists$ invertible $C \in M_n(\mathbb{R})$ so that $A = CBC^{-1}$).

$$\Updownarrow$$

A similar to B

$$C^{-1}AC = B$$

B similar to A

$$C^{-1}A(C^{-1})^{-1} = B$$

Theorem: Let $A, B \in M_n(\mathbb{R})$. Then A is similar to B iff A, B represent the same lin. trans. (in different bases), i.e. there are bases $\mathcal{B}, \mathcal{C}$ of $\mathbb{R}^n$ and a lin. trans. $T: \mathbb{R}^n \to \mathbb{R}^n$ so that

$$A = [T]_{\mathcal{B}}, \quad B = [T]_{\mathcal{C}}.$$

<u>Proof</u>: ($\Uparrow$) Did above.

($\Downarrow$) Given A, B w/ $A = CBC^{-1}$.

Define $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$T(x) = Ax$. Then $A = [T]_{\mathcal{E}}$.

Let $\mathcal{B} = \{$ cols. of $C \}$

$$\Rightarrow \quad C = [I]_{\mathcal{B}}^{\mathcal{E}}$$

$$\Rightarrow \quad C^{-1} = [I]_{\mathcal{E}}^{\mathcal{B}}$$

So

$$B = C^{-1} A C = [I]_{\mathcal{E}}^{\mathcal{B}} [T]_{\mathcal{E}} [I]_{\mathcal{B}}^{\mathcal{E}}$$

$$= [T]_{\mathcal{B}}$$

$\blacksquare$

Ex: Let

$$A = \begin{pmatrix} 13/2 & -1/2 & 3 \\ -19/2 & 7/2 & -6 \\ -9 & 1 & -4 \end{pmatrix}.$$

Let $T(x) = Ax.$ $\Rightarrow A = [T]_{\mathcal{E}}$

Let $\mathcal{B} = \left\{ \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ -5 \\ -2 \end{pmatrix} \right\}$

Compute $[T]_{\mathcal{B}}$.

$$[T]_{\mathcal{B}} = [I]^{\mathcal{B}}_{\mathcal{E}} [T]_{\mathcal{E}} [I]^{\mathcal{E}}_{\mathcal{B}}$$

$$= \left([I]^{\mathcal{E}}_{\mathcal{B}}\right)^{-1} A \, [I]^{\mathcal{E}}_{\mathcal{B}}$$

$$= \underbrace{\begin{pmatrix} -2 & 3 & 1 \\ 2 & 3 & -5 \\ 4 & -4 & -2 \end{pmatrix}^{-1}}_{C^{-1}} \begin{pmatrix} \sim\sim \end{pmatrix} \underbrace{\begin{pmatrix} -2 & 3 & 1 \\ 2 & 3 & -5 \\ 4 & -4 & -2 \end{pmatrix}}_{C}$$

$$\vdots$$
$$= \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix} = D$$

$\hookrightarrow$ Diagonal Matrix.

We have just diagonalized A.

Why diagonalize? For instance, suppose we needed to compute A^{10}.

We have:

$$C^{-1}AC = D$$
$$\Rightarrow A = CDC^{-1}$$

So

$$A^{10} = (CDC^{-1})^{10}$$

$$= C D \cancel{C^{-1}} \cdot \cancel{C} D \cancel{C^{-1}} \cancel{C} D \cancel{C^{-1}}$$

$$\cdots \cancel{C} D C^{-1}$$

$$= C D^{10} C^{-1}$$

$$= C \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}^{10} C^{-1}$$

$$= C \begin{pmatrix} 1^{10} & & \\ & 2^{10} & \\ & & 3^{10} \end{pmatrix} C^{-1}$$

$$= \cdots$$

<u>Q</u>: Why is A diagonalizable?

Let $\mathcal{B} = \{ v_1, v_2, v_3 \}$ so that $C = (v_1 \quad v_2 \quad v_3)$. Then

$$A v_1 = C D C^{-1} v_1$$

$$= C D e_1 = C e_1 = v_1$$

$$A v_2 = C D C^{-1} v_2$$

$$= C D e_2 = C(2e_2)$$

$$= 2(Ce_2) = 2v_2$$

$$A v_3 = \cdots\cdots = 3v_3$$

$\mathcal{B}$ is a special basis relative to A, it consists of eigenvectors of A.

Q: Where do eigenvectors come from? How can we compute them?

⋮

Soon..~