Subspaces

Let V be a v.s. and $H \subseteq V$.

vector space

We say H is a <u>subspace</u> of V if H is a vector space under the ops. on on V.

(H is a subspace)

<u>Thm</u>: Let V be a v.s, $H \subseteq V$

iff

1. $0 \in H$.
2. $v, w \in H \Rightarrow v+w \in H$.
3. $v \in H$, $c \in \mathbb{R} \Rightarrow cv \in H$.

<u>Moral</u>: One way to show a given set H is a v.s. is to realize it as a

subspace of some larger V that we already know is a v.s.

Then you only need to verify #1-3 of the Thm, instead of all 8 v.s. axioms.

Ex:

zero vector

1. $\{\vec{0}\} \leq V$, $V \leq V$

2. $\mathbb{R}[X] = \left\{ \begin{array}{l} \text{all polys. in } X \text{ w/ coeffs.} \\ \text{in } \mathbb{R} \end{array} \right\}$

$$\mathbb{P}_n = \{ f(X) \in \mathbb{R}[X] \mid \deg f \leq n \}$$

Claim: $\mathbb{P}_n \leq \mathbb{R}[X]$ ✓

1. $0 \in \mathbb{P}_n$ by def.n ($\deg 0 = -\infty$)

2. $f(X), g(X) \in \mathbb{P}_n \Rightarrow$

$$f(X) = \sum_{i=0}^{n} a_i X^i$$

$$+ \quad g(X) = \sum_{i=0}^{n} b_i X^i$$

$$f(X) + g(X) = \sum_{i=0}^{n} (a_i + b_i) X^i \in \mathbb{P}_n$$

3. Closure under scalar mult. is similar.

$\blacksquare$

Q: Do we still get a subspace if "$\le$" is replaced by "$=$"?

$$\mathbb{P}_n' = \{ f(X) \in \mathbb{R}[X] \mid \deg f = n \}$$

is $\cancel{\mathbb{P}_n' \le \mathbb{R}[X]}$? No

$n=2$:

$X^2, -X^2 \in \mathbb{P}_2'$,

but $x^2 + (-x^2) = 0 \notin \mathbb{P}_2'$

Similarly we can show $\mathbb{P}_n'$ is <u>not</u> a subspace of $\mathbb{R}[X]$.

3. $C^0((a,b)) = \{ f : (a,b) \to \mathbb{R} \mid f \text{ cont.} \}$

$\hookrightarrow$ or $[a,b]$

$C^0((a,b))$ is a subset of

$$V = \{ f : (a,b) \to \mathbb{R} \}$$

which is a v.s. So we can show $C^0((a,b))$ is a vector space by showing it is a subspace of V.

1. Yes, 0-fn. is cont., so it belongs to $C^0((a,b))$.

2. Since sums of cont. fns. are cont. (by Calc.), $C^0((a,b))$ is closed under add'n.

3. Likewise w/ scalar mult.

$\Rightarrow$ $\boxed{C^0((a,b)) \text{ is a subspace of } V}$

$C^1((a,b)) = \{f : (a,b) \to \mathbb{R} \mid f' \text{ is cont. on } (a,b)\}$

$C^2((a,b)) = \{f : (a,b) \to \mathbb{R} \mid f', f'' \text{ exist and are cont.}\}$

$\vdots$

$C^\infty((a,b)) = \{ \cdot - \cdot \mid f^{(n)} \text{ exists for all } n \geq 1\}$

As above, can show these are subspaces of V. Note we have

$$V \geq C^0((a,b)) \geq C^1((a,b)) \geq C^2((a,b)) \geq \cdots$$
$$\cdots \geq C^\infty((a,b))$$

4. Let H be all sequences:

$$\{a_1, 0, a_3, 0, a_5, 0, \ldots\}$$

inside

$$V = \{\text{all real sequences}\}.$$

$\underline{H \leq V}$:

1. $\{0, \underline{0}, 0, \underline{0}, \ldots\} \in H$?

2. $\{a_1, \underline{\underline{0}}, a_3, \underline{\underline{0}}, \ldots\} \in H$

$+ \ \{b_1, \underline{\underline{0}}, b_3, \underline{\underline{0}}, \ldots\} \in H$

$\text{sum} = \{a_1+b_1, \underline{\underline{0}}, a_3+b_3, \underline{\underline{0}}, \dots\} \in H$ ✓

3. $c\{a_1, 0, a_3, 0, \dots\}$

$c \cdot 0 = 0$

$= \{ca_1, 0, ca_3, 0, \dots\} \in H$

Span

Let V be a v.s. and $S \subseteq V$.
The span of S is

$$\text{Span}\, S = \left\{ \sum_{i=1}^{n} a_i v_i \;\middle|\; \begin{array}{l} a_i \in \mathbb{R} \\ v_i \in S \\ n \in \mathbb{N}_0 \end{array} \right\},$$

the set of all (finite) lin. combos. of elts. of S.

Remark: Usually we will assume $|S| < \infty$, but this restriction isn't necessary.

Thm: If V is a v.s. and $S \subseteq V$, then $\operatorname{Span} S \leq V$.

Proof: (Sketch) Must verify 3 subspace axioms.

1. $0 \in \operatorname{Span} S$? ✓

2. $v, w \in \operatorname{Span} S$

After inserting zero coeffs. as necessary

$$v = \sum_{i=1}^{n} a_i \underline{\underline{v_i}}, \quad a_i \in \mathbb{R}, \; v_i \in S$$

$$w = \sum_{i=1}^{n} b_i \underline{\underline{v_i}} \qquad b_j \in \mathbb{R}$$

$$v + w = \sum_{i=1}^{n} a_i v_i + \sum_{i=1}^{n} b_i v_i$$

Vector add. is comm.

$$= a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots a_n v_n$$
$$+ b_1 v_1 + b_2 v_2 + \cdots + b_n v_n$$
$$= (a_1 + b_1) v_1 + (a_2 + b_2) v_2 + \cdots$$

$\in S$

$\Rightarrow \in S$

So S is closed under vector add'n.

3. Scalar mult. is similar.

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