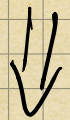


Linear Independence + Bases

Def: Let V be a V.S.,
 $S \subseteq V$. We say S is lin. ind. iff

$$\sum_i a_i v_i = 0 \quad w/ \quad \begin{array}{l} a_i \in \mathbb{R} \\ v_i \in S \end{array}$$



$$a_i = 0 \text{ for all } i$$

If lin. dep. = not lin. ind., then

S is lin. dep. iff \exists

$v_1, \dots, v_k \in S$ and $a_i \in \mathbb{R}$ not all

zero so that

$$\sum_{i=1}^k a_i v_i = 0.$$

"vector space"

"there exists"

Remark: $\{v\}$ is lin. dep. iff $v=0$

→ WLOG $a_1 \neq 0$, so

$$a_1 v_1 + \sum_{i>1} a_i v_i = 0$$

$$\underline{a_1} v_1 = - \sum_{i>1} a_i v_i$$

$\neq 0$

$$v_1 = \sum_{i>1} \left(\frac{-a_i}{a_1} \right) v_i$$

Theorem: $S \subseteq V$, $|S| \geq 2$, is
lin. dep. iff some $v \in S$ is
a lin. combo. of other $w \in S$:

$$v = \sum_{w \in S \setminus \{v\}} a_w w$$

Now assume $S = \{v_1, v_2, \dots, v_k\}$.

Can translate argument for \mathbb{R}^n to show:

Theorem: If $S \subseteq V$, $S = \{v_1, \dots, v_k\}$, $v_i \neq 0$, then S is lin. dep. iff $\exists 2 \leq j \leq k$ w/

$$v_j = \sum_{i < j} a_i v_i$$

↳ Lin. combo. of vectors before v_j .

Can also show:

Theorem: \forall v.s., $S \subseteq V$ lin. dep.

$$w/ \quad v = \sum_{w \neq v} a_w w \quad (w/ \quad v, w \in S).$$

Then

$$\text{Span } S = \text{Span}(S \setminus \{v\})$$

Def: V v.s., we say V is finitely generated if

$$V = \text{Span} \{v_1, \dots, v_k\}$$

for some $v_1, \dots, v_k \in V$.

Suppose $V = \text{Span} \{v_1, \dots, v_k\}$ is f.g.

We can assume $v_i \neq 0$ for all i .

By removing dependencies one at a time, we will eventually end up

w/ $\{v_1, \dots, v_k\}$ lin. ind.

Def: If V is a v.s. and $B \subseteq V$, we say B is a basis for V iff:

1. B is lin. ind.
2. $V = \text{Span } B$.

Work above shows:

Theorem: If V is a f.g. v.s., then V has a (finite) basis. In fact, any spanning set contains a basis.

Ex: 1. $\mathbb{P}_n = \{ f(x) \in \mathbb{R}[X] \mid \deg f \leq n \}$
 $= \{ a_0 + a_1 X + a_2 X^2 + \dots + a_n X^n \}$

$$= \text{Span} \{1, X, X^2, \dots, X^n\}$$

↙ ↘ \mathbb{P}_n is f.g.

Basis for \mathbb{P}_n

$$2. \mathbb{R}[X] = \{ \text{all polys in } X \}$$

$$= \text{Span} \{1, X, X^2, \dots\}$$

↙

Basis for $\mathbb{R}[X]$

$\Rightarrow \mathbb{R}[X]$ is not f.g.

$$3. V = \{ \text{all sequences in } \mathbb{R} \}$$

let

$$e_i = \{ 0, 0, \dots, 1, 0, \dots \}$$

↙
i-th term

Then

$$V \neq \text{Span} \{ e_1, e_2, \dots \}$$

Why?

$$\begin{aligned} & \{a_1, a_2, a_3, \dots\} \\ &= \{a_1, 0, 0, \dots\} + \{0, a_2, 0, \dots\} \\ & \quad + \{0, 0, a_3, 0, \dots\} + \dots \\ &= a_1 e_1 + a_2 e_2 + a_3 e_3 + \dots \end{aligned}$$

$\{e_1, e_2, e_3, \dots\}$ is not a basis for V . But it is a basis for

$$H = \left\{ \{a_n\} \mid a_n = 0 \text{ for } n \gg 1 \right\}$$

Remark: Can use set theory (specifically Zorn's lemma) to show that every v.s. has

a basis. In fact, any lin. ind.
subset can be "completed" to a
basis.