Linear Independence + Bases

"vector space"

Def: Let V be a v.s., $S \subseteq V$. We say S is lin. ind. iff

$$\sum_i a_i v_i = 0 \quad \text{w/} \quad \begin{array}{l} a_i \in \mathbb{R} \\ v_i \in S \end{array}$$

$$\Downarrow$$

$$a_i = 0 \text{ for all } i$$

If lin. dep. = not lin. ind., then S is lin. dep. iff $\exists$

"there exists"

$v_1, \ldots, v_k \in S$ and $a_i \in \mathbb{R}$ not all zero so that

$$\sum_{i=1}^{k} a_i v_i = 0.$$

Remark: $\{v\}$ is lin. dep. iff $v = 0$

WLOG $a_1 \neq 0$, so

$$a_1 v_1 + \sum_{i > 1} a_i v_i = 0$$

$$\underbrace{a_1}_{\neq 0} v_1 = - \sum_{i > 1} a_i v_i$$

$$v_1 = \sum_{i > 1} \left(\frac{-a_i}{a_1} \right) v_i$$

Theorem: $S \subseteq V$, $|S| \geq 2$, is lin. dep. iff some $v \in S$ is a lin. combo. of <u>other</u> $w \in S$:

$$v = \sum_{w \in S \setminus \{v\}} a_w w$$

Now assume $S = \{v_1, v_2, \ldots, v_k\}$.

Can translate argument for $\mathbb{R}^n$ to show:

<u>Theorem</u>: If $S \subseteq V$, $S = \{v_1, \ldots, v_k\}$, $v_1 \neq 0$, then S is lin. dep. iff $\exists\ 2 \leq j \leq k$ w/

$$v_j = \sum_{i<j} a_i v_i$$

$\hookrightarrow$ Lin. combo. of vectors <u>before</u> v_j.

Can also show:

<u>Theorem</u>: V v.s, $S \subseteq V$ lin. dep.

w/ $v = \sum_{w \neq v} a_w w$ (w/ $v, w \in S$).

Then

$$\mathrm{Span}\, S = \mathrm{Span}(S \setminus \{v\})$$

S w/o v.

Def: V v.s., we say V is ~~finitely~~ generated if

$$V = \mathrm{Span}\{v_1, \dots, v_k\}$$

for some $v_1, \dots, v_k \in V$.

Suppose $V = \mathrm{Span}\{v_1, \dots, v_k\}$ is f.g.

We can assume $v_i \neq 0$ for all i.

By removing dependencies one at a time, we will ~~eventually~~ end up w/ $\{v_1, \dots, v_k\}$ lin. ind.

Def: If V is a v.s. and $\mathcal{B} \subseteq V$, we say $\mathcal{B}$ is a **basis** for V iff:

1. $\mathcal{B}$ is lin. ind.
2. $V = \operatorname{Span} \mathcal{B}$.

Work above shows:

Theorem: If V is a f.g. v.s., then V has a (finite) basis. In fact, any spanning set contains a basis.

Ex: 1. $\mathbb{P}_n = \{ f(X) \in \mathbb{R}[X] \mid \deg f \leq n \}$

$$= \{ a_0 1 + a_1 X + a_2 X^2 + \cdots + a_n X^n \}$$

$= \mathrm{Span}\{1, X, X^2, \dots, X^n\}$

$\hookrightarrow \mathbb{P}_n$ <u>is</u> f.g.

Basis for $\mathbb{P}_n$

2. $\mathbb{R}[X] = \{$<u>all</u> polys. in $X\}$

$= \mathrm{Span}\{1, X, X^2, \dots\}$

Basis for $\mathbb{R}[X]$

$\Rightarrow \mathbb{R}[X]$ is <u>not</u> f.g.

3. $V = \{$all sequences in $\mathbb{R}\}$

Let

$e_i = \{0, 0, \dots, 1, 0, \dots\}$

$\llcorner$ ith term.

Then

$V \neq \mathrm{Span}\{e_1, e_2, \dots\}$

Why?

$$\{a_1, a_2, a_3, \ldots\}$$

$$= \{a_1, 0, 0, \ldots\} + \{0, a_2, 0, \ldots\}$$

$$+ \{0, 0, a_3, 0, \ldots\} + \cdots$$

$$= a_1 e_1 + a_2 e_2 + a_3 e_3 + \cdots$$

$\{e_1, e_2, e_3, \ldots\}$ is not a basis for V. But it is a basis for

$$H = \{\{a_n\} \mid a_n = 0 \text{ for } n \gg 1\}$$

Remark: Can use set theory (specifically Zorn's lemma) to show that every v.s. has a basis. In fact, any lin. ind. subset can be "completed" to a basis.