# Eigenvectors & Eigenvalues

Let $T: V \to V$ be a lin. trans. of v.s. We say $v \in V$ is an <u>eigenvector</u> of $T$ iff:

- $v \neq 0$
- $T(v) = \lambda v$ for some $\lambda \in \mathbb{R}$

$\lambda$ is called the <u>eigenvalue</u> assoc. to $v$.

<u>Notice</u>: $T(v) = \lambda v \iff T(v) - \lambda v = 0$

$$\iff (T - \lambda I)(v) = 0$$

$$\iff \boxed{v \in \ker(T - \lambda I)}$$

So if $v \neq 0$ then $\ker(T - \lambda I) \neq \{0\}$.

If $V$ is f.d. (i.e. is $\mathbb{R}^n$), then

$\ker(T - \lambda I) \neq \{0\} \iff T - \lambda I$ is *not* inv.

$\iff \det(T - \lambda I) = 0$

$\iff \boxed{\det(\lambda I - T) = 0}$

Characteristic eqn. of $T$.

**Moral**: To find eits. of $T$, must solve $\det(\lambda I - T) = 0$ for $\lambda$.

Given an eit. $\lambda$, to find the corr. eigenvectors, must compute

$$\boxed{E_\lambda = \ker(T - \lambda I) = \ker(\lambda I - T)}$$

by row reduction of $T - \lambda I$.

The *$\lambda$-eigenspace* of $T$.

Ex: Find eigenvalues/vectors of

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Char. Eqn.:

$$0 = |\lambda I - A| = \left| \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right|$$

$$= \begin{vmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{vmatrix} = (\lambda-1)(\lambda-4) - 6$$

$$\boxed{0 = \lambda^2 - 5\lambda - 2}$$

$$\Rightarrow \quad \lambda = \frac{5 \pm \sqrt{25 + 8}}{2}$$

$$\boxed{\lambda = \frac{5 \pm \sqrt{33}}{2}} \text{ (Eigenvalues)}$$

<u>Eigenvectors</u>: We need to compute $\ker(A - \lambda I)$ for <u>each</u> $\lambda$ above.

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{pmatrix}$$ ← <u>Must</u> be singular.

$$\downarrow \text{RR}$$

$$\begin{pmatrix} 3 & 4-\lambda \\ 0 & 0 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & \frac{4-\lambda}{3} \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad x_1 + \frac{4-\lambda}{3} x_2 = 0$$

$$\Rightarrow \quad x_1 = \frac{\lambda-4}{3} x_2$$

$$\Rightarrow \quad \underset{\text{eigenvector}}{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{\lambda - 4}{3} x_2 \\ x_2 \end{pmatrix}$$

$$= x_2 \begin{pmatrix} (\lambda - 4)/3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad E_\lambda = \text{Span} \left\{ \begin{pmatrix} (\lambda - 4)/3 \\ 1 \end{pmatrix} \right\}$$

$$= \text{Span} \left\{ \begin{pmatrix} \lambda - 4 \\ 3 \end{pmatrix} \right\}$$

$$\left( \lambda = \frac{5 \pm \sqrt{33}}{2} \right)$$

$$= \text{Span} \left\{ \begin{pmatrix} \frac{5 \pm \sqrt{33}}{2} - 4 \\ 3 \end{pmatrix} \right\}$$

$$= \text{Span} \left\{ \begin{pmatrix} 5 \pm \sqrt{33} - 8 \\ 6 \end{pmatrix} \right\}$$

$$\boxed{E_\lambda = \operatorname{Span}\left\{\begin{pmatrix} -3 \pm \sqrt{33} \\ 6 \end{pmatrix}\right\}}$$

Eigenspaces.

Check:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -3+\sqrt{33} \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 9+\sqrt{33} \\ 3(5+\sqrt{33}) \end{pmatrix}$$

equal?

$$\lambda = \frac{5+\sqrt{33}}{2}$$

$$= \frac{5+\sqrt{33}}{2}\begin{pmatrix} \frac{2(9+\sqrt{33})}{5+\sqrt{33}} \\ 6 \end{pmatrix} \rightarrow = -3+\sqrt{33}$$

✓

$$\frac{2(9+\sqrt{33})}{5+\sqrt{33}} = \frac{2(9+\sqrt{33})}{5+\sqrt{33}} \cdot \frac{5-\sqrt{33}}{5-\sqrt{33}}$$

$$= \frac{2(45 - 33 + 5\sqrt{33} - 9\sqrt{33})}{25 - 33}$$

$$= \frac{2(12 - 4\sqrt{33})}{-8} = \frac{8(3 - \sqrt{33})}{-8}$$

$$= -3 + \sqrt{33}$$

**Theorem**: If $A$ is a triangular matrix (upper or lower), then the eigs. of $A$ are its diagonal entries.

**Proof**: If $A$ is upper $\Delta$, then char. eqn. is

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - a_{11} & & & \\ & \lambda - a_{22} & & \\ & & \ddots & \\ & 0 & & \lambda - a_{nn} \end{vmatrix}$$

$$= (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn})$$

$\Rightarrow$ sol'ns (e'vls.) are $a_{11}, a_{22}, \ldots$ $\boxtimes$

**Ex**: Find eigenvalues/dimensions of eigenspaces for

$$A = \begin{pmatrix} -3 & 1 & 0 & 2 \\ & -3 & -1 & 2 \\ & & 2 & 4 \\ & & & -1 \end{pmatrix}$$

Upper $\Delta$, so sol'ns of char. are:

$$\lambda = \underbrace{-3, -3}_{\text{Alg. multiplicity } 2}, \underbrace{2, -1}_{\text{Alg. mult. } 1}$$

Note that char. eqn. is

$$(\lambda+3)^2(\lambda-2)(\lambda+1) = 0$$

## Eigenspaces: (Want dim. only)

$\underline{\lambda = -3}$:

$$-3I - A = \begin{pmatrix} 0 & -1 & 0 & -3 \\ & 0 & 1 & -2 \\ & & -5 & -4 \\ & & & -2 \end{pmatrix}$$

$$\xrightarrow{RR} \begin{pmatrix} 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$$

$$\xrightarrow{RR} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ has $\underline{3}$ pivots

$$\Rightarrow \boxed{\dim \ker(-3I - A) = 1}$$

↳ dim. of $-3$ eigenspace

$\underline{\lambda = 2}$ :

$$2I - A = \begin{pmatrix} 5 & -1 & 0 & -3 \\ & 5 & 1 & -2 \\ & & 0 & -4 \\ & & & 3 \end{pmatrix}$$

$$\xrightarrow{RR} \begin{pmatrix} \boxed{5} & -1 & 0 & 0 \\ 0 & \boxed{5} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ has 3 pivots

$$\Rightarrow \boxed{\dim(\ker(2I - A)) = 1}$$

↳ dim. of 2-eigenspace of $A$.

$\underline{\lambda = -1}$ : Similar. We find

$$\boxed{\dim(\ker(-I - A)) = 1}$$

↳ dim. of $(-1)$-eigenspace.

$\blacksquare$

**Theorem**: If $A \in M_n(\mathbb{R})$, then $|\lambda I - A|$ is a polynomial in $\lambda$ of degree $n$. We call this the **char. poly.** of $A$.

**Proof**: It should be clear that $|\lambda I - A|$ is a poly. in $\lambda$. That $|\lambda I - A|$ has degree $n$ in $\lambda$ comes from looking more closely at the determinant $\boxtimes$

**Cor**: Eigenvalues of $A$ are roots of its char. poly.

**Cor.**: If $A \in M_n(\mathbb{R})$, then $A$ has at *most* $n$ distinct eigenvalues.

**Def**: The multiplicity of an eigenvalue $\lambda$ in the char. poly. of $A \in M_n(\mathbb{R})$ is called the *algebraic multiplicity* of $\lambda$. The quantity

$$\dim E_\lambda = \dim \ker(\lambda I - A)$$

is the *geometric multiplicity* of $\lambda$.

**Theorem**: Let $A \in M_n(\mathbb{R})$ w/ distinct eits. $\lambda_1, \lambda_2, \ldots, \lambda_k$.

If $v_i$ is a $\lambda_i$-eigenvector, then $\{v_1, v_2, \ldots, v_k\}$ is *lin. ind.*

( Eigenvectors w/ distinct eigenvalues
are lin. ind. )

<u>Proof</u> : Next time ...