Eigenvectors & Diagonalizability

Recall: Given $A \in M_n(\mathbb{R})$, a vector $v \in \mathbb{R}^n$ is an <u>eigenvector</u> of A iff:

- $v \neq 0$
- $Av = \lambda v$

<u>eigenvalue</u>

The eigenvalues of A can be found by solving the <u>char. eq.</u>:

$$\underbrace{\det(\lambda I - A)}_{\substack{\text{char. poly. of } A, \\ \text{has degree } n.}} = 0$$

To find eigenvectors for a given λ, we must compute

$$E_\lambda(A) = E_\lambda = \ker(\lambda I - A)$$

λ-eigenspace of A

Theorem: Let $A \in M_n(\mathbb{R})$ w/ distinct eigenvalues $\lambda_1, \dots, \lambda_k$. If v_i is a λ_i-eigenvector, then $\{v_1, v_2, \dots, v_k\}$ is lin. ind.

Proof: Suppose not. Then

$$v_j = \sum_{i<j} c_i v_i \qquad (*)$$

for some $c_i \in \mathbb{R}$, $j \leq k$. Choose

j as small as possible. Mult. $(*)$ by A and λ_j:

$$Av_j = \sum_{i<j} c_i A v_i$$

$$\lambda_j v_j = \sum_{i<j} c_i \lambda_i v_i \qquad v_i \text{ is a } \lambda_i\text{-eigenv.}$$

$$-\left(\lambda_j v_j = \sum_{i<j} c_i \lambda_j v_i\right)$$

$$\boxed{0 = \sum_{i<j} c_i(\lambda_i - \lambda_j) v_i}$$

Since j is as small as possible, we know $\{v_1, v_2, \ldots, v_{j-1}\}$ is lin. ind.

$$c_i \underbrace{(\lambda_i - \lambda_j)}_{\neq 0} = 0 \text{ for all } i<j$$

$\Rightarrow c_i = 0$ for all $i<j$.

So $(*) \Rightarrow v_j = 0$ ~~→←~~

(since we know $v_j \neq 0$)

$\blacksquare$

<u>Goal</u>: Given $A \in M_n(\mathbb{R})$ (which we think of as $T: \mathbb{R}^n \to \mathbb{R}^n$ given by $T(x) = Ax$), find a basis $\mathcal{B}$ of $\mathbb{R}^n$ for which $[T]_{\mathcal{B}}$ is diagonal (all off-diag. entries $= 0$).

That is, find a diag. matrix D that is similar to A:

$C = (\text{vectors in } \mathcal{B})$

$$A = CDC^{-1}$$

D: diag.

Suppose $v_1, v_2, \dots, v_k$ are lin. ind. eigenvectors of A w/ eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ (not nec. distinct).

Complete to a basis for $\mathbb{R}^n$:

$$[\lambda_2 v_2]_{\mathcal{B}} = \lambda_2 [v_2]_{\mathcal{B}} = \lambda_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \lambda_2 e_2$$

$$\mathcal{B} = \{ v_1, v_2, \dots, v_k, u_1, \dots, u_\ell \}$$

w/ $k + \ell = n$

What is $[T]_{\mathcal{B}}$?

$$[T]_{\mathcal{B}} = \Big([T(v_1)]_{\mathcal{B}} \, [T(v_2)]_{\mathcal{B}} \cdots [T(v_k)]_{\mathcal{B}} \; [T(u_1)]_{\mathcal{B}} \cdots [T(u_\ell)]_{\mathcal{B}} \Big)$$

$$= \Big([Av_1]_{\mathcal{B}} \; [Av_2]_{\mathcal{B}} \cdots [Av_k]_{\mathcal{B}}$$

$$\cdots \; [Au_1]_{\mathcal{B}} \; \cdots \; [Au_\ell]_{\mathcal{B}} \Big)$$

$$= \Big([\lambda_1 v_1]_{\mathcal{B}} \; [\lambda_2 v_2]_{\mathcal{B}} \cdots [\lambda_k v_k]_{\mathcal{B}} \; [Au_1]_{\mathcal{B}} \cdots [Au_\ell]_{\mathcal{B}} \Big)$$

$$= \Big(\lambda_1 e_1 \; \lambda_2 e_2 \cdots \lambda_k e_k \; \cancel{\ast} \Big)$$

$$= \left(\begin{array}{c|c} \begin{matrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_k \end{matrix} & \cancel{\ast} \\ \hline & \end{array}\right)$$

$k \times k$

$\ell \times \ell$

Conversely, if $[T]_{\mathcal{B}} =$, then

$$[T]_{\mathcal{B}}\, e_i = \lambda_i e_i \quad (i=1,\dots,k)$$

So:

cols. are members of $\mathcal{B}$

$$A = C\,[T]_{\mathcal{B}}\,C^{-1}$$

$$C = (v_1 \cdots v_k\ u_1 \cdots u_\ell)$$

$$A v_i = C[T]_{\mathcal{B}}\underbrace{C^{-1} v_i}_{e_i}$$

$$= C[T]_{\mathcal{B}}\, e_i \qquad (1 \le i \le k)$$

$$= C\lambda_i e_i = \lambda_i (C e_i)$$

$$= \lambda_i v_i$$

$\Rightarrow$ v_i is a λ_i-eigenvector of A for $i=1,\dots,k$.

Theorem: $A \in M_n(\mathbb{R})$ is diagonalizable iff there is a basis of $\mathbb{R}^n$ whose members are all eigenvectors of A (<u>A-eigenbasis</u>).

Since eigenvectors from different eigenspaces are lin. ind., the above happens iff

$$\boxed{n = \dim E_{\lambda_1} + \dim E_{\lambda_2} + \cdots + \dim E_{\lambda_k}}$$

where $\lambda_1, \ldots, \lambda_k$ are <u>distinct</u> eits. of A.

<u>Ex</u>:

1. Let $A = \begin{pmatrix} -10 & 9 & -9 \\ 0 & -1 & 0 \\ 12 & -12 & 11 \end{pmatrix}$.

Is A diagonalizable?

$$\det(\lambda I - A) = \lambda^3 - 3\lambda - 2$$

rational root test

$$= (\lambda+1)^2(\lambda-2)$$

$\Rightarrow$ Eigenvalues are $\lambda = -1, 2$

$$E_{-1} = \ker(-I - A) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$$E_2 = \ker(2I - A) = \operatorname{Span}\left\{ \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} \right\}$$

So $\dim E_{-1} + \dim E_2 = 2 + 1 = 3$

$= \dim \mathbb{R}^3$

$\Rightarrow$ A <u>is</u> diagonalizable.

In fact:

$$A = \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 0 & -1 & -4 \end{pmatrix}}_{\substack{\underbrace{\quad\quad}_{-1} \ \underbrace{\ }_{2}}} \begin{pmatrix} -1 & & \\ & -1 & \\ & & 2 \end{pmatrix} \overbrace{\begin{pmatrix} -4 & 5 & -3 \\ 4 & -4 & 3 \\ -1 & 1 & -1 \end{pmatrix}}^{\text{inv.}}$$

2. $$A = \begin{pmatrix} 2 & -9 & 0 \\ -2 & 2 & 1 \\ 12 & -27 & -4 \end{pmatrix}$$

$$\det(\lambda I - A) = \lambda^3 - 3\lambda - 2$$
$$= (\lambda+1)^2(\lambda-2)$$

$\Rightarrow$ Eigenvalues are $\lambda = -1, 2$

$$E_{-1} = \ker(-I - A) = \mathrm{Span}\left\{\begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}\right\}$$

$$E_2 = \ker(2I - A) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right\}$$

$$\Rightarrow \dim E_{-1} + \dim E_2 = 1 + 1$$
$$= 2 \neq 3$$

$\Rightarrow$ $\boxed{A \text{ is } \underline{\text{not}} \text{ diagonalizable}}$

<u>Comment</u>: The closest to diagonal we can make A via similarity is:

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Q: So when can we have

$$n = \sum_{\lambda} \dim E_{\lambda}$$

λ runs over the eigenvalues of A } spectrum of A

Note: If $A = CBC^{-1}$, then:

$$\text{char. poly. of } A = \det(\lambda I - A)$$

$$= \det(\lambda I - CBC^{-1})$$

$$= \det(\lambda C I C^{-1} - CBC^{-1})$$

$$= \det(C(\lambda I - B)C^{-1})$$

$$= \underline{\det(C)} \cdot \det(\lambda I - B) \cdot \underline{\det(C)^{-1}}$$

$$= \det(\lambda I - B) = \text{char. poly. of } B$$

Similar matrices have the **same** char. polys.