Linear Algebra - Final Exam

When: 5/18 ← Due → Tues.

5/15 ← Released → Sat.

What: Cumulative

OH: Posted to webpage...

Diagonalization (Cont.)

Let $A \in M_n(\mathbb{R})$. If λ is an eigenvalue of A:

$$E_\lambda(A) = \lambda\text{-eigenspace}$$

(A) → optional

$$= \mathrm{Null}(\lambda I - A) \neq \{0\}$$

$\longrightarrow$ A is similar to diag. matr.

Theorem: A is diagonalizable iff

$$\sum_{\lambda \leftarrow \text{all eigenvalues of } A} \dim E_\lambda(A) = n,$$

i.e. there is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

Last time we showed: similar matrices have the same char. poly.

Let $0 \neq v \in E_\lambda(A)$. Suppose A is similar to B:

$$A = CBC^{-1}$$

$$Av = (CBC^{-1})v$$

$$\lambda v = C(BC^{-1}v)$$

$$C^{-1}(\lambda v) = B(C^{-1}v)$$

$$\lambda \cdot C^{-1}v = B(C^{-1}v)$$

So $0 \neq C^{-1}v \in E_\lambda(B)$. That is, mult. by C^{-1} is a lin. trans.

$$E_\lambda(A) \longrightarrow E_\lambda(B)$$

$$v \longmapsto C^{-1}v$$

$$Cw \longleftarrow\!\!\shortmid\ w$$

Thus C^{-1} (or C) provides an iso. between $E_\lambda(A)$, $E_\lambda(B)$. Thus

$$\dim E_\lambda(A) = \dim E_\lambda(B),$$

i.e. dimensions of eigenspaces of similar matrices are the same

<u>Ex</u>: Let

$$A = \begin{pmatrix} 2 & -9 & 0 \\ -2 & 2 & 1 \\ 12 & -27 & -4 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & 1 & \\ & -1 & \\ & & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & -5 & 1 \\ 1 & -2 & 0 \\ 3 & -3 & 2 \end{pmatrix}.$$

You can check that $A = CBC^{-1}$. So:

$$\begin{tikzcd} E_\lambda(A) \arrow[r, bend left, "C^{-1}v"] & E_\lambda(B) \arrow[l, bend left, "Cv"] \end{tikzcd}$$

Direct computation shows

$$E_{-1}(B) = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$$

So:

mult. by C

$$E_{-1}(A) = \operatorname{Span}\left\{\begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}\right\} \checkmark$$

Likewise

$$E_2(B) = \operatorname{Span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

mult. by C

$$E_2(A) = \operatorname{Span}\left\{\begin{pmatrix}1\\0\\2\end{pmatrix}\right\} \checkmark$$

Now suppose $\dim E_\lambda(A) = k > 0$.

Let $\{v_1, v_2, \dots, v_k\}$ be a basis for $E_\lambda(A)$. Find a basis

$$\mathcal{B} = \{v_1, \dots, v_k, u_1, \dots, u_\ell\}$$

for $\mathbb{R}^n$. Then if $T(x) = Ax$:

$$[T]_{\mathcal{B}} = \Big([T(v_1)]_{\mathcal{B}}\ [T(v_2)]_{\mathcal{B}} \cdots [T(v_k)]_{\mathcal{B}}$$

$$[T(u_1)]_{\mathcal{B}} \cdots [T(u_\ell)]_{\mathcal{B}} \Big)$$

$$= \Big([\lambda v_1]_{\mathcal{B}}\ [\lambda v_2]_{\mathcal{B}} \cdots [\lambda v_k]_{\mathcal{B}}$$

$$[T(u_1)]_{\mathcal{B}} \cdots [T(u_\ell)]_{\mathcal{B}} \Big)$$

$$XI - B = \left(\begin{array}{cccc|ccc} X-\lambda & & & & & & \\ & X-\lambda & & & & & \\ & & \ddots & & & & \\ & & & X-\lambda & & & \\ \hline & & & & X & & \\ & & & & & \ddots & \\ & & & & & & X \end{array}\right) = B$$

$k \times k$

Since A and B are similar:

$$\text{char. poly. } A = \text{char. poly. } B$$

$$= \det(XI - B)$$

$$= \boxed{(X-\lambda)^k}\, \underbrace{g}_{\text{some other poly.}}(X)$$

So

$$\dim E_\lambda(A) = k \leq \text{mult. of } k \text{ as a root of char. poly.}$$

$$\text{Geom. mult. of } \lambda \leq \text{Alg. mult. of } \lambda$$

$$\underbrace{}\qquad\qquad\underbrace{}$$

$$\gamma(\lambda) \qquad\qquad \alpha(\lambda)$$

Theorem: If $A \in M_n(\mathbb{R})$ w/ eigenvalue λ, then

$$\gamma(\lambda) \leq \alpha(\lambda).$$

$$n = \sum_{\lambda} \gamma(\lambda) \leq \sum_{\lambda} \alpha(\lambda) = \text{deg. of poly.} = n$$

when A is diag.

Cor. A is diag. iff $\gamma(\lambda) = \alpha(\lambda)$ for <u>every</u> eigenvalue of A.

Cor.: If char. poly. of A does not have repeated roots, then

A is diagonalizable. (A has exactly n eigenvalues)

Proof: In this case $\alpha(\lambda)=1$ for all λ. Thus:

$$1 \leq \gamma(\lambda) \leq \alpha(\lambda) = 1$$

Eigenspaces always have dim. ≥ 1

$$\Rightarrow \gamma(\lambda) = \alpha(\lambda) \text{ for}$$

all eits. λ of A. ✓

$\boxed{\text{a}}$

Ex: Let

$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 3 \end{pmatrix}$$

Then

$$\det(\lambda I - A) = \lambda^3 - 6\lambda^2 + 11\lambda - 6$$

Roots can be found using rat. root test

$$= (\lambda-1)(\lambda-2)(\lambda-3)$$

Since A is 3×3 and has 3 eigenvalues,

<u>A is diagonalizable</u>.

$\boxtimes$