Matrix Algebra

Ryan C. Daileda

Trinity University

Linear Algebra

Recall

Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ has the form $T(\mathbf{x}) = A\mathbf{x}$, where A is the standard matrix

$$A = [T] = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{pmatrix}.$$

We defined matrix multiplication to correspond to composition of linear transformations:

$$[S][T] = [S \circ T].$$

In terms of columns, this is given by

$$AB = A \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{pmatrix} = \begin{pmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{pmatrix},$$

i.e. the columns of AB are found by applying A to the columns of B.

Warnings

Although matrix multiplication has many familiar algebraic properties, there are several notable differences.

Even though AB and BA may both be defined and have the same dimensions, in general $AB \neq BA$.

For instance, if

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then

$$AB = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{but} \quad BA = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

If $AB = BA$, we say that A and B *commute* with each other.

It also possible to have $AB = 0$ even though *both* A and B are nonzero.

For instance, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

More generally, matrix multiplication *does not* obey the cancellation law:

$$AB = AC \;\; \not\Rightarrow \;\; B = C.$$

Matrix Powers

If A is a square matrix and $k \in \mathbb{N}$, we define

$$A^k = \underbrace{A \cdots A}_{k \text{ times}}.$$

This makes sense since:

- The product of two $n \times n$ matrices is again an $n \times n$ matrix.
- Matrix multiplication is associative.

If we define

$$A^0 = I,$$

then matrix powers obey the usual laws of exponents (as long as the exponents aren't negative).

The Transpose of a Matrix

If A is an $m \times n$ matrix, its *transpose* is the $n \times m$ matrix A^T whose columns are formed by "standing up" the rows of A.

For example:

$$A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & -1 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \\ 0 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -3 & 1 \\ 0 & 1 & 4 \end{pmatrix} \Rightarrow B^T = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & 1 \\ -1 & 1 & 4 \end{pmatrix}$$

Remarks.

- In terms of matrix entries,

$$(A^T)_{ij} = A_{ji}.$$

- If we treat a vector $\mathbf{v} \in \mathbb{R}^n$ as an $n \times 1$ matrix, then $\mathbf{v}^T$ is a $1 \times n$ *row vector*.

The transpose interacts nicely with matrix arithmetic.

Theorem 1 (Properties of the Transpose)

For compatible matrices A and B, and any scalar $c \in \mathbb{R}$:

a. $(A^T)^T = A$
b. $(A + B)^T = A^T + B^T$
c. $(cA)^T = c(A^T)$
d. $(AB)^T = B^T A^T$

Although properties (a)-(c) are fairly intuitive, (d) takes a little more thought.

Using the row-column rule for matrix multiplication, we have

$$((AB)^T)_{ij} = (AB)_{ji} = \sum_k A_{jk} B_{ki} = \sum_k (A^T)_{kj} (B^T)_{ik}$$

$$= \sum_k (B^T)_{ik} (A^T)_{kj} = (B^T A^T)_{ij}.$$

It follows that $(AB)^T = B^T A^T$.

Remark. Although the transpose may seem like a somewhat arbitrary operation, it can be interpreted in terms of linear transformations via *linear functionals*.

Matrix Inverses

Definition

An $n \times n$ matrix A is called *invertible* if there is an $n \times n$ matrix B so that

$$AB = BA = I.$$

In this case we call B the *inverse* of A and write $B = A^{-1}$.

Remarks.

- Only square matrices can have inverses, but not every square matrix is invertible.
- The inverse of a matrix is the analogue of the reciprocal of a real number.

Example

If

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix},$$

then

$$AB = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

and

$$BA = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

It follows that $B = A^{-1}$.

Remark. Strictly speaking, to show that $B = A^{-1}$ one must show that *both* $AB = I$ and $BA = I$. It turns out, though, that either of these equations actually implies the other!

Inverses of 2×2 Matrices

The following result can be extremely useful.

Theorem 2

If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \textit{and} \quad ad - bc \neq 0,$$

then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The proof is by a straightforward computation and is left as an exercise.

Inverses and Solutions of Linear Equations

To solve the (ordinary) equation $5x = 3$, we multiply both sides by $1/5 = 5^{-1}$ to obtain $x = 3/5$.

This procedure has a perfect analogue for matrix equations involving invertible matrices.

Theorem 3

If A is an invertible $n \times n$ matrix, then for any $\mathbf{b} \in \mathbb{R}^n$ the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof. It is easy to see that $\mathbf{x} = A^{-1}\mathbf{b}$ is indeed a solution:

$$A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b}.$$

On the other hand, if we know that $A\mathbf{x} = \mathbf{b}$, then

$$A^{-1}\mathbf{b} = A^{-1}(A\mathbf{x}) = (A^{-1}A)\mathbf{x} = I\mathbf{x} = \mathbf{x}.$$

□

Remark. Note how the proof of Theorem 3 utilizes the "two-sided" nature of A^{-1}.

Example 1

Use matrix inversion to solve the linear system

$$\begin{aligned} 8x_1 + 6x_2 &= 2, \\ 5x_1 + 4x_2 &= -1. \end{aligned}$$

Solution. The given system is equivalent to the matrix equation

$$\begin{pmatrix} 8 & 6 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Because $8 \cdot 4 - 5 \cdot 6 = 2 \neq 0$, the coefficient matrix is invertible, so the solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 & 6 \\ 5 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 4 & -6 \\ -5 & 8 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 14 \\ -18 \end{pmatrix} = \begin{pmatrix} 7 \\ -9 \end{pmatrix}.$$

□

Fundamental Questions

Given A^{-1}, solving $A\mathbf{x} = \mathbf{b}$ is very easy.

This leads us to two important questions:

- How can we tell if a square matrix A is invertible?
- If we know A is invertible, how do we compute A^{-1}?

We will return to these questions next time!