Matrix Inversion

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Linear Algebra

Recall

Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ has the form $T(\mathbf{x}) = A\mathbf{x}$, where A is the standard matrix

$$A = [T] = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{pmatrix}, \ \mathbf{e}_j = (\delta_{ij}).$$

We defined our matrix operations to correspond to the addition, scalar multiplication and composition of linear transformations:

$$[S] + [T] = [S + T],$$
$$c[S] = [cS],$$
$$[S][T] = [S \circ T].$$

In terms of matrix columns:

$$AB = A \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{pmatrix} = \begin{pmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{pmatrix}.$$

Warnings

Although matrix multiplication has many familiar algebraic properties, there are several notable differences.

Even though AB and BA may both be defined and have the same dimensions, in general $AB \neq BA$.

For instance, if

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then

$$AB = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{but} \quad BA = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

If $AB = BA$, we say that A and B *commute* with each other.

It also possible to have $AB = 0$ even though *both* A and B are nonzero.

For instance, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

More generally, matrix multiplication *does not* obey the cancellation law:

$$AB = AC \quad \not\Rightarrow \quad B = C.$$

Matrix Powers

If A is a square matrix and $k \in \mathbb{N}$, we define

$$A^k = \underbrace{A \cdots A}_{k \text{ times}}.$$

This makes sense since:

- The product of two $n \times n$ matrices is again an $n \times n$ matrix.
- Matrix multiplication is associative.

If we define

$$A^0 = I,$$

then matrix powers obey the usual laws of exponents (as long as the exponents aren't negative).

The Transpose of a Matrix

If A is an $m \times n$ matrix, its *transpose* is the $n \times m$ matrix A^T whose columns are formed by "standing up" the rows of A.

For example:

$$A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & -1 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \\ 0 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -3 & 1 \\ 0 & 1 & 4 \end{pmatrix} \Rightarrow B^T = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & 1 \\ -1 & 1 & 4 \end{pmatrix}$$

Remarks.

- In terms of matrix entries,

$$(A^T)_{ij} = A_{ji}.$$

- If we treat a vector $\mathbf{v} \in \mathbb{R}^n$ as an $n \times 1$ matrix, then $\mathbf{v}^T$ is a $1 \times n$ *row vector*.

The transpose interacts nicely with matrix arithmetic.

Theorem 1 (Properties of the Transpose)

For compatible matrices A and B, and any scalar $c \in \mathbb{R}$:

a. $(A^T)^T = A$
b. $(A+B)^T = A^T + B^T$
c. $(cA)^T = c(A^T)$
d. $(AB)^T = B^T A^T$

Although properties (a)-(c) are fairly intuitive, (d) takes a little more thought.

Using the row-column rule for matrix multiplication, we have

$$((AB)^T)_{ij} = (AB)_{ji} = \sum_k A_{jk}B_{ki} = \sum_k (A^T)_{kj}(B^T)_{ik}$$
$$= \sum_k (B^T)_{ik}(A^T)_{kj} = (B^T A^T)_{ij}.$$

It follows that $(AB)^T = B^T A^T$.

Remark. Although the transpose may seem like a somewhat arbitrary operation, it can be interpreted in terms of linear transformations via *linear functionals*.

Matrix Inverses

Definition

An $n \times n$ matrix A is called *invertible* if there is an $n \times n$ matrix B so that

$$AB = BA = I.$$

In this case we call B the *inverse* of A and write $B = A^{-1}$.

Remarks.

- Only square matrices can have inverses, but not every square matrix is invertible!
- The inverse of a matrix is the analogue of the reciprocal of a real number.
- The inverse of a square matrix (if it exists) is unique, since

$$AB = CA = I \;\Rightarrow\; C = CI = C(AB) = (CA)B = IB = B.$$

Example

If

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix},$$

then

$$AB = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

and

$$BA = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

It follows that $B = A^{-1}$.

Remark. Strictly speaking, to show that $B = A^{-1}$ one must show that *both* $AB = I$ and $BA = I$. It turns out, though, that either of these equations actually implies the other!

Inverses of 2×2 Matrices

The following result can be extremely useful.

Theorem 2

If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad ad - bc \neq 0,$$

then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The proof is by a straightforward computation and is left as an exercise.

Inverses and Solutions of Linear Equations

To solve the (ordinary) equation $5x = 3$, we multiply both sides by $1/5 = 5^{-1}$ to obtain $x = 3/5$.

This procedure has a perfect analogue for matrix equations involving invertible matrices.

Theorem 3

If A is an invertible $n \times n$ matrix, then for any $\mathbf{b} \in \mathbb{R}^n$ the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof. It is easy to see that $\mathbf{x} = A^{-1}\mathbf{b}$ is indeed a solution:

$$A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b}.$$

On the other hand, if we know that $A\mathbf{x} = \mathbf{b}$, then

$$A^{-1}\mathbf{b} = A^{-1}(A\mathbf{x}) = (A^{-1}A)\mathbf{x} = I\mathbf{x} = \mathbf{x}.$$

$\square$

Remark. Note how the proof of Theorem 3 utilizes the "two-sided" nature of A^{-1}.

Example 1

Use matrix inversion to solve the linear system

$$8x_1 + 6x_2 = 2,$$
$$5x_1 + 4x_2 = -1.$$

Solution. The given system is equivalent to the matrix equation

$$\begin{pmatrix} 8 & 6 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Because $8 \cdot 4 - 5 \cdot 6 = 2 \neq 0$, the coefficient matrix is invertible, so the solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 & 6 \\ 5 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 4 & -6 \\ -5 & 8 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 14 \\ -18 \end{pmatrix} = \begin{pmatrix} 7 \\ -9 \end{pmatrix}.$$

□

Properties of Inversion

The following properties are easy consequences of the uniqueness of the matrix inverse.

Theorem 4

Let A, B be $n \times n$ matrices.

a. *If A is invertible, then A^{-1} is invertible and*

$$(A^{-1})^{-1} = A.$$

b. *If any two of A, B and AB is invertible, then so is the third, and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

c. *If A is invertible, then so is A^T and*

Proof (Sketch)

To see why (b) is true, suppose A and B are invertible. Then

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I,$$
$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

This shows that AB is invertible and that $(AB)^{-1} = B^{-1}A^{-1}$.

Parts (a) and (c) are proved in a similar fashion.

WLOG now suppose that A and AB are invertible.

Then A^{-1} is invertible by part (a) and $A^{-1}(AB)$ is invertible by our work above.

But $A^{-1}(AB) = (A^{-1}A)B = IB = B$, so B is invertible, too.

Therefore all three of A, B and AB are invertible, and $(AB)^{-1} = B^{-1}A^{-1}$ follows as above.

This finishes the proof of part (c). □

Remarks.

- Where did we use the “two-sided” nature of the inverse in the proof above?

- Part (b) generalizes to

$$(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_1^{-1}.$$

Fundamental Questions

Given A^{-1}, solving $A\mathbf{x} = \mathbf{b}$ is very easy.

This leads us to two important questions:

- How can we tell if a square matrix A is invertible?
- If we know A is invertible, how do we compute A^{-1}?

We turn to Theorem 3 for guidance. If A is invertible, then $A\mathbf{x} = \mathbf{b}$ has a solution for all possible $\mathbf{b}$.

This means that A must have a pivot in every row, and hence every column, too, since A is square.

Sufficient Conditions for Invertibility

These conditions actually guarantee invertibility as well.

To see this, suppose that A is a square matrix with a pivot in every row (and column).

Then for each j the equation $A\mathbf{x} = \mathbf{e}_j$ has a solution, $\mathbf{x} = \mathbf{b}_j$.

Let $B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix}$. We then have

$$AB = \begin{pmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{pmatrix} = I.$$

Notice that if $B\mathbf{x} = \mathbf{0}$, then

$$\mathbf{x} = I\mathbf{x} = (AB)\mathbf{x} = A(B\mathbf{x}) = A\mathbf{0} = \mathbf{0}.$$

That is, the only solution to $B\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

This means B must have a pivot in every column (and row, since B is square).

The same reasoning then shows that there is a square matrix C so that $BC = I$.

We then have

$$A = AI = A(BC) = (AB)C = IC = C.$$

Thus $BA = BC = I$. So we have

$$AB = I = BA,$$

which shows that A is invertible and $A^{-1} = B$.

The Invertible Matrix Theorem - Version 1

We have therefore proven:

Theorem 5 (The Invertible Matrix Theorem)

For a square matrix A, TFAE:

a. *A is invertible.*
b. *A has a pivot in each column.*
c. *A has a pivot in each row.*
d. *The equation $A\mathbf{x} = \mathbf{0}$ only has the solution $\mathbf{x} = \mathbf{0}$.*
e. *The equation $A\mathbf{x} = \mathbf{b}$ has a (unique) solution for all $\mathbf{b}$.*

Moreover, in this case, if $\mathbf{b}_j$ is the solution to $A\mathbf{x} = \mathbf{e}_j$, then

$$A^{-1} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix}.$$

Computing A^{-1}

We have now seen that the columns of A^{-1} are the solutions to $A\mathbf{x} = \mathbf{e}_j$.

To compute these solutions we need to row reduce the augmented matrix $\begin{pmatrix} A & \mathbf{e_j} \end{pmatrix}$, for $j = 1, 2, \ldots, n$.

We can perform these computations *simultaneously* by row reducing the "super augmented" matrix

$$\begin{pmatrix} A & \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{pmatrix} = \begin{pmatrix} A & I \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} I & A^{-1} \end{pmatrix}.$$

This gives us an algorithm for computing A^{-1}.

Computation of A^{-1} Through Row Reduction

Let's record this important result.

Theorem 6

If A is an invertible square matrix, then the row operations that transform A to the identity matrix will transform the identity matrix into A^{-1}. That is,

$$\begin{pmatrix} A & I \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} I & A^{-1} \end{pmatrix}.$$

Remark. *Cramer's rule* gives an explicit formula for the entries in A^{-1} in terms of determinants, which we will study in Chapter 3.

Example

Let

$$A = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 2 & -1 & 3 & 1 \\ 0 & -3 & -1 & 2 \\ 3 & 1 & -1 & 0 \end{pmatrix}.$$

We have

$$\begin{pmatrix} 1 & -2 & 0 & 3 & 1 & 0 & 0 & 0 \\ 2 & -1 & 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & -3 & -1 & 2 & 0 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{2}{7} \\ 0 & 1 & 0 & 0 & \frac{25}{56} & -\frac{11}{56} & -\frac{4}{7} & -\frac{1}{56} \\ 0 & 0 & 1 & 0 & \frac{1}{56} & \frac{13}{56} & -\frac{1}{7} & -\frac{9}{56} \\ 0 & 0 & 0 & 1 & \frac{19}{28} & -\frac{5}{28} & -\frac{3}{7} & -\frac{3}{28} \end{pmatrix}$$

This simultaneously shows that A is invertible (why?) and tells us that

$$A^{-1} = \begin{pmatrix} -\frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{2}{7} \\ \frac{25}{56} & -\frac{11}{56} & -\frac{4}{7} & -\frac{1}{56} \\ \frac{1}{56} & \frac{13}{56} & -\frac{1}{7} & -\frac{9}{56} \\ \frac{19}{28} & -\frac{5}{28} & -\frac{3}{7} & -\frac{3}{28} \end{pmatrix}.$$

Remarks.

- Note that although the entries in A were integers, the same is *not* true of A^{-1}.
- However, the denominators in A^{-1} are all divisors of 56. Cramer's rule will explain this phenomenon.

Coordinate Maps Revisited

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for $\mathbb{R}^n$.

Recall that the $\mathcal{B}$-coordinate map $[\cdot]_{\mathcal{B}} : \mathbb{R}^n \to \mathbb{R}^n$ is defined implicitly by the equation

$$\underbrace{\begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix}}_{B} [\mathbf{x}]_{\mathcal{B}} = \mathbf{x}.$$

Because the columns of B are linearly independent, B is invertible. Hence

$$\boxed{[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix}^{-1} \mathbf{x}}.$$

Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and we know the values $T(\mathbf{b}_i)$ for $i = 1, 2, \ldots, n$.

To compute T from this information we notice that

$$\begin{aligned} T(\mathbf{x}) &= [T]\mathbf{x} = [T]BB^{-1}\mathbf{x} \\ &= \begin{pmatrix} [T]\mathbf{b}_1 & [T]\mathbf{b}_2 & \cdots & [T]\mathbf{b}_n \end{pmatrix} B^{-1}\mathbf{x} \\ &= \begin{pmatrix} T(\mathbf{b}_1) & T(\mathbf{b}_2) & \cdots & T(\mathbf{b}_n) \end{pmatrix} B^{-1}\mathbf{x}. \end{aligned}$$

This proves:

Theorem 7

If $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, then the standard matrix for T is given by

$$[T] = \begin{pmatrix} T(\mathbf{b}_1) & \cdots & T(\mathbf{b}_n) \end{pmatrix} \begin{pmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{pmatrix}^{-1}.$$

Example

Let L be a line through the origin in $\mathbb{R}^2$ and define $T : \mathbb{R}^2 \to \mathbb{R}^2$ to be reflection through L.

Every line through the origin in $\mathbb{R}^2$ can be given by an equation of the form $ax + by = 0$ with

$$\mathbf{n} = \begin{pmatrix} a \\ b \end{pmatrix} \neq \mathbf{0}.$$

If we rotate $\mathbf{n}$ by 90 degrees we get

$$\mathbf{v} = \begin{pmatrix} -b \\ a \end{pmatrix},$$

which is clearly not a multiple of $\mathbf{n}$. Therefore $\mathcal{B} = \{\mathbf{n}, \mathbf{v}\}$ is a basis for $\mathbb{R}^2$.

Because **v** is parallel to L while **n** is perpendicular to it, we have

$$T(\mathbf{v}) = \mathbf{v}, \quad T(\mathbf{n}) = -\mathbf{n}.$$

It follows from Theorem 7 that

$$\begin{aligned}
[T] &= \begin{pmatrix} \mathbf{v} & -\mathbf{n} \end{pmatrix} \begin{pmatrix} \mathbf{v} & \mathbf{n} \end{pmatrix}^{-1} = \begin{pmatrix} -b & -a \\ a & -b \end{pmatrix} \begin{pmatrix} -b & a \\ a & b \end{pmatrix}^{-1} \\
&= \frac{-1}{a^2+b^2} \begin{pmatrix} -b & -a \\ a & -b \end{pmatrix} \begin{pmatrix} b & -a \\ -a & -b \end{pmatrix} \\
&= \frac{-1}{a^2+b^2} \begin{pmatrix} a^2-b^2 & 2ab \\ 2ab & b^2-a^2 \end{pmatrix}.
\end{aligned}$$

Let's record our conclusion.

Theorem 8

The standard matrix for reflection through the line with equation $ax + by = 0$ *is*

$$[T] = \frac{-1}{a^2 + b^2}\begin{pmatrix} a^2 - b^2 & 2ab \\ 2ab & b^2 - a^2 \end{pmatrix}.$$

Examples.

- Reflection through the x-axis ($y = 0 \Rightarrow a = 0, b = 1$) is given by

$$\frac{-1}{1}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

as we saw earlier.

- Reflection through the line $y = x$ ($x - y = 0$ $\Rightarrow a = 1, b = -1$) is given by

$$\frac{-1}{2}\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

  in agreement with earlier work.

Given a line through the origin in $\mathbb{R}^2$ with equation $ax + by = 0$, we are free to scale the normal vector $\mathbf{n} = (a, b)$ as we see fit.

In particular, we may assume that $\mathbf{n}$ is a unit vector, i.e. $a^2 + b^2 = 1$. This means that (a, b) lies on the unit circle.

Therefore there is an angle θ so that

$$a = \cos\theta \quad \text{and} \quad b = \sin\theta.$$

The double angle formulas then give

$$a^2 - b^2 = \cos^2\theta - \sin^2\theta = \cos 2\theta,$$
$$2ab = 2\cos\theta\sin\theta = \sin 2\theta.$$

The reflection matrix therefore can be written

$$\frac{-1}{a^2+b^2}\begin{pmatrix} a^2-b^2 & 2ab \\ 2ab & b^2-a^2 \end{pmatrix} = -\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$
$$= -\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This shows that every reflection in $\mathbb{R}^2$ is simply reflection through the y-axis, followed by an appropriate rotation.

In the other direction, if we are given an angle ϕ and we set

$$a = \cos\frac{\phi}{2} \quad \text{and} \quad b = \sin\frac{\phi}{2},$$

then reflection through the line $ax + by = 0$ is given by

$$[T] = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = [T] \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

which shows that every rotation is the composition of two reflections!