The Invertible Matrix Theorem

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Linear Algebra

Introduction

It is important to recognize when a square matrix is invertible.

We can now characterize invertibility in terms of every one of the concepts we have now encountered.

We will continue to develop criteria for invertibility, adding them to our list as we go.

The invertibility of a matrix is also related to the invertibility of linear transformations, which we discuss below.

Theorem 1 (The Invertible Matrix Theorem)

For a square $(n \times n)$ matrix A, TFAE:

a. *A is invertible.*
b. *A has a pivot in each row/column.*
c. $A \xrightarrow{RREF} I$.
d. *The equation $A\mathbf{x} = \mathbf{0}$ only has the solution $\mathbf{x} = \mathbf{0}$.*
e. *The columns of A are linearly independent.*
f. $\operatorname{Null} A = \{\mathbf{0}\}$.
g. *A has a left inverse ($BA = I_n$ for some B).*
h. *The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one to one.*
i. *The equation $A\mathbf{x} = \mathbf{b}$ has a (unique) solution for any $\mathbf{b}$.*
j. $\operatorname{Col} A = \mathbb{R}^n$.
k. *A has a right inverse ($AC = I_n$ for some C).*
l. *The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.*
m. *A^T is invertible.*

Inverse Transforms

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ (also called an *endomorphism* of $\mathbb{R}^n$) is called *invertible* iff it is both one-to-one and onto.

If $[T]$ is the standard matrix for T, then we know T is given by

$$\mathbf{x} \mapsto [T]\mathbf{x}.$$

The Invertible Matrix Theorem tells us that this transformation is invertible iff $[T]$ is invertible.

In this case, let $B = [T]^{-1}$ and define $S : \mathbb{R}^n \to \mathbb{R}^n$ by $S(\mathbf{x}) = B\mathbf{x}$.

Notice that

$$S(T(\mathbf{x})) = B([T]\mathbf{x}) = (B[T])\mathbf{x} = I\mathbf{x} = \mathbf{x},$$
$$T(S(\mathbf{x})) = [T](B\mathbf{x}) = ([T]B)\mathbf{x} = I\mathbf{x} = \mathbf{x}.$$

That is

$$S \circ T = T \circ S = \mathrm{Id},$$

where $\mathrm{Id} : \mathbb{R}^n \to \mathbb{R}^n$ is the *identity transformation* given by

$$\mathrm{Id}(\mathbf{x}) = \mathbf{x} \quad \text{or} \quad [\mathrm{Id}] = I.$$

We call S the *inverse* of T and write $S = T^{-1}$. That is,

$$[T^{-1}] = [T]^{-1}.$$

This reasoning can be reversed. If

$$S \circ T = T \circ S = \mathrm{Id},$$

then

$$I = [\mathrm{Id}] = [S \circ T] = [T \circ S] = [S][T] = [T][S],$$

which shows that $[T]$ is invertible and $[T]^{-1} = [S]$. Thus:

Theorem 2

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then TFAE:

- *T is invertible.*
- *$[T]$ is invertible.*
- *There is a linear transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ so that $S \circ T = T \circ S = \mathrm{Id}$.*

Remarks

We don't always have to consider the conditions "one-to-one" and "onto" simultaneously.

It is not hard to argue that we have:

Theorem 3

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then:

a. *T is one to one iff there is a linear transformation $S : \mathbb{R}^m \to \mathbb{R}^n$ to that $S \circ T = \mathrm{Id}$ (T has a left inverse).*
b. *T is onto iff there is a linear transformation $R : \mathbb{R}^m \to \mathbb{R}^n$ to that $T \circ R = \mathrm{Id}$ (T has a right inverse).*

We leave the proof as an exercise.