Linear Algebra: Day 1

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Linear Algebra

Introduction

Linear algebra is concerned with the study of systems of linear equations and the abstractions (matrix algebra, vector spaces, etc.) that can be used to understand them.

Linear systems and vector spaces arise in a host of applications, such as integration, differential equations, quadratic forms, representation theory, etc.

We will focus on both the computational and theoretical aspects of linear algebra, which means that you will be asked to become adept at both matrix manipulations as well as elementary mathematical proofs.

Linear Systems

A linear equation in the variables $x_1, x_2, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where b and the *coefficients* a_i are all real (or complex) numbers.

Remarks.

- In practice, n can be extremely large, but we will usually limit ourselves to $n \leq 5$.
- There is no standard terminology for the constant b. We will simply call it the *constant* or the *RHS* of the equation.
- The vast majority of the theorems we will encounter apply just as well if we require the constants to lie in a given *field* (e.g. the rational numbers).

Examples

The equation

$$2x_1 - x_2 + \frac{1}{3}x_3 + \sqrt{5}x_4 = 7$$

is linear in 4 variables with real coefficients.

The equation

$$(1 + 2i)x_1 + 2x_3 + (4 + 5i)x_3 = 3i$$

is linear in 3 variables with complex coefficients.

The equation $3x_2 + x_3 = 4x_1 - x_3$ is also linear, since it can be rewritten as

$$4x_1 - 3x_2 - 2x_3 = 0.$$

Linear Systems

An $m \times n$ *linear system* consists of m simultaneous linear equations in n variables $x_1, x_2, \ldots, x_n$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

Note that a_{ij} is the coefficient of x_j in the ith equation.

So the first index of a coefficient specifies the *equation*, while the second index specifies *where* in the equation it occurs.

Examples

The following is a 2×3 linear system:

$$\begin{aligned} -x_1 + 2x_2 - 3x_3 &= 4, \\ 5x_1 - 6x_2 + 7x_3 &= 8. \end{aligned}$$

Here's a 4×3 system:

$$\begin{aligned} x_1 + x_2 \phantom{{}+x_3} &= 1, \\ x_1 \phantom{{}+x_2} - x_3 &= -1, \\ x_2 + x_3 &= 0, \\ x_1 - x_2 + x_3 &= 7. \end{aligned}$$

"Missing" variables simply have coefficient zero, i.e. we have $a_{13} = 0$, $a_{22} = 0$, $a_{31} = 0$.

Solutions of Linear Systems

A *solution* to a linear system in n variables is a tuple $(s_1, s_2, \ldots, s_n)$ of numbers so that setting $x_1 = s_1$, $x_2 = s_2, \ldots, x_n = s_n$ makes *every* equation in the system true.

Example. $(10, 7, 0)$ is a solution to

$$\begin{aligned} -x_1 + 2x_2 - 3x_3 &= 4, \\ 5x_1 - 6x_2 + 7x_3 &= 8. \end{aligned}$$

So is $(9, 5, -1)$.

We will say that two linear systems in n variables are *equivalent* if they have exactly the same solutions.

Gaussian Elimination

We can find all the solutions of a linear system S (if solutions exist) by repeatedly replacing S with equivalent systems S_1, $S_2, \ldots, S_k$ until we can "read off" the solutions.

The idea is to systematically use each equation to "cancel" variables in the equations below it. This process is called *Gaussian elimination*.

Example 1

Solve the linear system

$$\begin{aligned} x_2 + 5x_3 &= -4, \\ x_1 + 4x_2 + 3x_3 &= -2, \\ 2x_1 + 7x_2 - x_3 &= 1. \end{aligned}$$

Solution. We start by interchanging the first two equations. This clearly results in an equivalent system:

$$\begin{aligned} x_2 + 5x_3 &= -4 \\ x_1 + 4x_2 + 3x_3 &= -2 \\ 2x_1 + 7x_2 - x_3 &= 1 \end{aligned} \iff \begin{aligned} x_1 + 4x_2 + 3x_3 &= -2 \\ x_2 + 5x_3 &= -4 \\ 2x_1 + 7x_2 - x_3 &= 1. \end{aligned}$$

Now subtract twice the first equation from the third to eliminate x_1. Because we can undo this operation by instead *adding* twice the first equation, this results in the equivalent system

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= -2 \\ x_2 + 5x_2 &= -4 \\ -x_2 - 7x_3 &= 5. \end{aligned}$$

Now add the second equation to the third. Again, this is reversible, so we get an equivalent system.

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= -2 \\ x_2 + 5x_3 &= -4 \\ -x_2 - 7x_3 &= 5 \end{aligned} \iff \begin{aligned} x_1 + 4x_2 + 3x_3 &= -2 \\ x_2 + 5x_3 &= -4 \\ -2x_3 &= 1. \end{aligned}$$

Divide the third equation by -2:

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= -2 \\ x_2 + 5x_3 &= -4 \\ x_3 &= -1/2. \end{aligned}$$

This system is *triangular* and is easily solved by back substitution, but we press on.

Add appropriate multiples of the third equation to the previous two in order to eliminate x_3:

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= -2 \\ x_2 + 5x_3 &= -4 \\ x_3 &= -1/2 \end{aligned} \iff \begin{aligned} x_1 + 4x_2 &= -1/2 \\ x_2 &= -3/2 \\ x_3 &= -1/2. \end{aligned}$$

Finally, add -4 times the second equation to the first to obtain:

$$\begin{aligned} x_1 &= 11/2 \\ x_2 &= -3/2 \\ x_3 &= -1/2. \end{aligned}$$

The only solution to this system is obviously $(11/2, -3/2, -1/2)$, and since all of our steps are reversible, we conclude that

$$\boxed{(11/2, -3/2, -1/2)}$$

is the *unique* solution to our original system. □

Coefficient and Augmented Matrices

In Gaussian elimination, the variables and equals signs merely act as placeholders. All of the "action" happens by combining corresponding coefficients appropriately.

So, given a linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

we extract all of the constants and put them into a pair of matrices.

The *coefficient matrix* of the system is

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

and its *augmented matrix* is

$$(A \quad \mathbf{b}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix} \quad \text{where} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

i.e. we obtain the augmented matrix by adding a column of b_i's to the coefficient matrix.

Row Operations

We can solve a linear system via Gaussian elimination by simply performing the corresponding operations on the rows of its augmented matrix.

Example 2

Solve the linear system

$$\begin{aligned} x_1 \phantom{{}+2x_2} - 3x_3 &= 8, \\ 2x_1 + 2x_2 + 9x_3 &= 7, \\ x_2 + 5x_3 &= -2. \end{aligned}$$

Solution. We introduce the augmented matrix and simultaneously perform Gaussian elimination on it and the system itself.

We put the system on the left and the corresponding augmented matrix on the right. Denote the ith equation by E_i and the ith row of the augmented matrix by R_i.

$$\begin{array}{rrrcr} x_1 & & -3x_3 & = & 8 \\ 2x_1 & +2x_2 & +9x_3 & = & 7 \\ & x_2 & +5x_3 & = & -2 \end{array} \qquad \begin{pmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{pmatrix}$$

$$[-2E_1 + E_2 \to E_2] \qquad [-2R_1 + R_2 \to R_2]$$

$$\begin{array}{rrrcr} x_1 & & -3x_3 & = & 8 \\ & 2x_2 & +15x_3 & = & -9 \\ & x_2 & +5x_3 & = & -2 \end{array} \qquad \begin{pmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{pmatrix}$$

$$[E_2 \leftrightarrow E_3] \qquad [R_2 \leftrightarrow R_3]$$

$$\begin{array}{rrrr} x_1 & & -3x_3 & = & 8 \\ & x_2 & +5x_3 & = & -2 \\ & 2x_2 & +15x_3 & = & -9 \end{array} \qquad \begin{pmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 2 & 15 & -9 \end{pmatrix}$$

$[-2E_2 + E_3 \to E_3]$ $\qquad$ $[-2R_2 + R_3 \to R_3]$

$$\begin{array}{rrrr} x_1 & & -3x_3 & = & 8 \\ & x_2 & +5x_3 & = & -2 \\ & & +5x_3 & = & -5 \end{array} \qquad \begin{pmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 5 & -5 \end{pmatrix}$$

$[(1/5)E_3 \to E_3]$ $\qquad$ $[(1/5)R_3 \to R_3]$

$$\begin{array}{rrrr} x_1 & & -3x_3 & = & 8 \\ & x_2 & +5x_3 & = & -2 \\ & & x_3 & = & -1 \end{array} \qquad \begin{pmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$[-5E_3 + E_2 \to E_2] \qquad [-5R_3 + R_2 \to R_2]$$

$$[3E_3 + E_1 \to E_1] \qquad [3R_3 + R_1 \to R_1]$$

$$\begin{aligned} x_1 &= 5 \\ x_2 &= 3 \\ x_3 &= -1 \end{aligned} \qquad \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

The 1's to the left of the "augmented column" indicate that we've solved for each variable, and that we have the unique solution

$$\boxed{(5, 3, -1)}.$$

□

Row Reduction

Passage to the augmented matrix and back yields a 1-1 correspondence:

$$\boxed{m \times n \text{ linear systems}} \longleftrightarrow \boxed{m \times (n+1) \text{ matrices}}.$$

Note that we could have solved the preceding problem simply by converting to the augmented matrix, performing row operations and then converting back to equations at the end.

Performing Gaussian elimination at the level of augmented matrices is called *row reduction*.

Elementary Row Operations

The row operations that produce matrices representing equivalent linear systems are called *Elementary Row Operations*.

Elementary Row Operations

- (Replacement) Replace one row by itself plus a multiple of another row.
- (Interchange) Interchange two rows.
- (Scaling) Multiply all entries in a row by a *nonzero* constant.

Remarks.

- These are the *only allowable row operations* during row reduction of a matrix!
- Note that every elementary row operation is reversible.

Fundamental Questions

Now that we have an efficient algorithm for solving linear systems (row reduction of augmented matrices), we pose two important questions.

Given a linear system:

1. (Existence) Does the system have *any* solutions?
2. (Uniqueness) If there is a solution, is it unique, i.e. is it the *only one*?

Systems that have solutions are called *consistent*. Systems with no solutions are called *inconsistent*.

More Examples

Example 3

Use row reduction to show that the system

$$\begin{array}{rrrrcl} x_1 & -6x_2 & & & = & 5 \\ & x_2 & -4x_3 & +x_4 & = & 0 \\ -x_1 & +6x_2 & +x_3 & +5x_4 & = & 3 \\ & -x_2 & +5x_3 & +4x_4 & = & 0 \end{array}$$

is inconsistent.

Solution. We pass to the augmented matrix and row reduce:

$$\begin{pmatrix} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ -1 & 6 & 1 & 5 & 3 \\ 0 & -1 & 5 & 4 & 0 \end{pmatrix} \xrightarrow{R_1+R_3 \to R_3} \begin{pmatrix} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & -1 & 5 & 4 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & -1 & 5 & 4 & 0 \end{pmatrix} \xrightarrow{R_2+R_4\to R_4} \begin{pmatrix} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 1 & 5 & 0 \end{pmatrix}$$

$$\xrightarrow{-R_3+R_4\to R_4} \begin{pmatrix} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & -8 \end{pmatrix}.$$

The final row corresponds to the equation $0 = -8$, which is clearly impossible. Therefore the original system must be inconsistent. □

Example 4

Use row reduction to show that the system

$$\begin{array}{rrrrcl} x_1 & -6x_2 & & & = & 5 \\ & x_2 & -4x_3 & +x_4 & = & 0 \\ -x_1 & +6x_2 & +x_3 & +5x_4 & = & 3 \\ & -x_2 & +5x_3 & +4x_4 & = & 8 \end{array}$$

is consistent. Do not determine the solution(s).

Solution. We pass to the augmented matrix and row reduce:

$$\begin{pmatrix} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ -1 & 6 & 1 & 5 & 3 \\ 0 & -1 & 5 & 4 & 0 \end{pmatrix} \xrightarrow{R_1+R_3 \to R_3} \begin{pmatrix} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & -1 & 5 & 4 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & -1 & 5 & 4 & 8 \end{pmatrix} \xrightarrow{R_2+R_4\to R_4} \begin{pmatrix} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 1 & 5 & 8 \end{pmatrix}$$

$$\xrightarrow{-R_3+R_4\to R_4} \begin{pmatrix} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[6R_2+R_1\to R_1]{4R_3+R_2\to R_2} \begin{pmatrix} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The final row corresponds to the equation $0 = 0$, which is always true and therefore redundant.

The other rows show that given *any* value of x_4, we can easily solve for x_1, x_2 and x_3. So the original system is consistent (with infinitely many solutions!). □