Vectors

Ryan C. Daileda

Trinity University

Linear Algebra

Introduction

Today we will introduce vectors and vector arithmetic in $\mathbb{R}^n$.

As we will see, we can represent systems of *multiple* equations as *single* vector equations.

In particular, we can use vector arithmetic to represent linear systems as equations involving *linear combinations* of vectors.

This naturally leads to the notion of the *span* of a set of vectors, and gives us a new way to talk about the existence of solutions to linear systems.

Vectors

Given a positive integer n, an $n \times 1$ matrix is called a *(column) vector.*

Thus, a vector has the form

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix},$$

where each v_i is a real (or complex) number.

We let $\mathbb{R}^n$ (or $\mathbb{C}^n$) denote the set of all such vectors.

We will write $\mathbf{v} \in \mathbb{R}^n$ to indicate that $\mathbf{v}$ is a vector in $\mathbb{R}^n$.

So, for example, we have:

$$\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ -3/2 \end{pmatrix} \in \mathbb{R}^2,$$

$$\begin{pmatrix} 1/5 \\ 7 \\ \pi \end{pmatrix}, \begin{pmatrix} -6 \\ \sqrt{7} \\ 4 \end{pmatrix} \in \mathbb{R}^3,$$

$$\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \in \mathbb{R}^4,$$

etc.

As with matrices, we say that two vectors are *equal* provided their corresponding entries are equal.

That is, if

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix},$$

are vectors in $\mathbb{R}^n$, then

$$\mathbf{u} = \mathbf{v} \iff u_i = v_i \text{ for every } i.$$

Remark. We cannot directly compare vectors with different numbers of entries.

Vector Arithmetic

We define the *sum* of two vectors in $\mathbb{R}^n$ to be

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{pmatrix},$$

i.e. we add two vectors by adding their corresponding entries.

Given a vector $\mathbf{v} \in \mathbb{R}^n$, we define the *scalar multiple* of $\mathbf{v}$ by a real number $c \in \mathbb{R}$ to be

$$c\mathbf{v} = c \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{pmatrix}.$$

That is, we scale a vector $\mathbf{v}$ by a real number c by multiplying every entry of $\mathbf{v}$ by c.

Given a vector $\mathbf{v} \in \mathbb{R}^n$, we define its *negative* to be

$$-\mathbf{v} = (-1)\mathbf{v},$$

which is simply the vector whose entries are the negatives of those in $\mathbf{v}$. We define *vector subtraction* by

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

We define the *zero vector* to be

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n.$$

Strictly speaking, there is a different zero vector for each n, but we will use the same symbol for all of them.

Algebraic Properties of Vectors

Vector arithmetic inherits many familiar properties from the ordinary arithmetic of real numbers.

Specifically, it is not hard to argue that:

Theorem 1 (Algebraic Properties of Vectors)

For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *and any scalars* $c, d \in \mathbb{R}$ *we have:*

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$
4. $\mathbf{v} + (-\mathbf{v}) = (-\mathbf{v}) + \mathbf{v} = \mathbf{0}$
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$
7. $c(d\mathbf{v}) = (cd)\mathbf{v}$
8. $1\mathbf{v} = \mathbf{v}$

Other algebraic properties can be deduced from these. For instance, since $0 + 0 = 0$:

$$0\mathbf{v} = (0 + 0)\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}.$$

Thus

$$\begin{aligned} \mathbf{0} &= 0\mathbf{v} + (-0\mathbf{v}) \\ &= (0\mathbf{v} + 0\mathbf{v}) + (-0\mathbf{v}) \\ &= 0\mathbf{v} + (0\mathbf{v} + (-0\mathbf{v})) \\ &= 0\mathbf{v} + \mathbf{0} = 0\mathbf{v}. \end{aligned}$$

That is, $0\mathbf{v} = \mathbf{0}$.

This can, of course, be derived directly from the definition of scalar multiplication, but this approach will be useful later on when we deal with abstract vector spaces.

Linear Combinations

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ and scalars $c_1, c_2, \ldots, c_k \in \mathbb{R}$, the vector

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

is called the *linear combination* of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ with *weights* $c_1, c_2, \ldots, c_k$.

Remark. Technically, addition is only defined between *pairs* of vectors. But the associative property of vector addition, namely

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}),$$

can be used to show that an arbitrary (finite) sum of vectors is the same regardless of how parentheses are inserted.

Vector Equations and Linear Systems

We can use vector equations involving linear combinations to represent linear systems.

For instance, consider the linear system

$$\begin{aligned} 2x_1 \qquad\quad - x_3 &= -3, \\ x_1 + 5x_2 + 3x_3 &= 1, \\ - x_2 + x_3 &= 2. \end{aligned}$$

This is the same as the *single* vector equation

$$\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_3 \\ x_1 + 5x_2 + 3x_3 \\ -x_2 + x_3 \end{pmatrix} = x_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 5 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}.$$

In general, a linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{21}x_2 + \cdots + a_{2n}x_n &= b_2, \\ \vdots \qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

with coefficient matrix and "constant vector"

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

is equivalent to the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}.$$

Put another way, the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

is equivalent to the system with augmented matrix

$$\begin{pmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b}\end{pmatrix}.$$

Thus, a linear system has a solution iff the "constant vector" **b** is a linear combination of the columns of the coefficient matrix.

Example

Example 1

If

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \ \mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \ \mathbf{a}_3 = \begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix},$$

determine whether or not $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$.

Solution. The augmented matrix of the corresponding linear system is

$$\begin{pmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since the reduced echelon form *does not* have a pivot in the last column, this system is consistent, which means that

$$\boxed{\mathbf{b} \text{ is a linear combination of } \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3.}$$

In fact, since the parametric form of the solution set is

$$\begin{aligned} x_1 &= 2 - 5x_3, \\ x_2 &= 3 - 4x_2, \\ x_3 &\text{ is free,} \end{aligned}$$

we can find specific weights that yield $\mathbf{b}$ simply by choosing x_3. So, for example, with $x_3 = 2$ we have

$$\mathbf{b} = -8\mathbf{a}_1 - 5\mathbf{a}_2 + 2\mathbf{a}_3.$$

$\square$

Span

Definition

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ (which need not be distinct), we define

$$\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$$

to be the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

Thus, a vector $\mathbf{b}$ belongs to $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ if and only if the system with augmented matrix

$$\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k & \mathbf{b} \end{pmatrix}$$

is consistent. In this case we write

$$\mathbf{b} \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}.$$

Example

In Example 1 we showed that if

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \ \mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \ \mathbf{a}_3 = \begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix},$$

then $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$.

Thus $\mathbf{b} \in \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. Since

$$\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_k,$$

we always have

$$\mathbf{0} \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}.$$

Furthermore, since

$$\mathbf{v}_i = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 1\mathbf{v}_i + \cdots + 0\mathbf{v}_k,$$

we also have

$$\mathbf{v}_i \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \quad \text{for all } i.$$

Even more generally, suppose that $\mathbf{v}, \mathbf{w} \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ and that $c \in \mathbb{R}$ is a scalar.

Then, by the definition of the span, we can find weights so that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k,$$
$$\mathbf{w} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_k\mathbf{v}_k.$$

Because vector addition is commutative, this means that

$$\begin{aligned}\mathbf{v} + \mathbf{w} &= (c_1\mathbf{v}_1 + d_1\mathbf{v}_1) + (c_2\mathbf{v}_2 + d_2\mathbf{v}_2) + \cdots + (c_k\mathbf{v}_k + c_k\mathbf{v}_k)\\ &= (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2 + \cdots + (c_k + d_k)\mathbf{v}_k,\end{aligned}$$

which also belongs to $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

Also, because scalar multiplication is distributive and associative:

$$\begin{aligned} c\mathbf{v} &= c(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) \\ &= c(c_1\mathbf{v}_1) + c(c_2\mathbf{v}_2) + \cdots + c(c_k\mathbf{v}_k) \\ &= (cc_1)\mathbf{v}_1 + (cc_2)\mathbf{v}_2 + \cdots + (cc_k)\mathbf{v}_k, \end{aligned}$$

which belongs to $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, too.

Because of these properties, one says that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is *closed* under vector addition and scalar multiplication.

Remark. This proves that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a *subspace* of $\mathbb{R}^n$ (a term we will define later).

The Geometry of Vectors

We can visualize vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ as arrows, exactly as in Calculus III.

In these cases, the addition law we have defined here is precisely the usual “tip to tail” rule.

And scalar multiplication has the same “scaling” effect.

We can use these facts to give a geometric description of the span (in low dimensions).

Span$\{\mathbf{v}\}$

If $\mathbf{v} \neq \mathbf{0}$ belongs to $\mathbb{R}^2$ or $\mathbb{R}^3$, then Span$\{\mathbf{v}\}$ consists of all the scalar multiples of $\mathbf{v}$.

These are all the vectors that have the same (or opposite) direction as $\mathbf{v}$.

If we draw our vectors with their tails at the origin, this means that Span$\{\mathbf{v}\}$ can be visualized as the line through the origin with direction given by $\mathbf{v}$.

Because of this, we will sometimes refer to Span$\{\mathbf{v}\}$ as the "line through $\mathbf{v}$," even when working in $\mathbb{R}^n$ in general.

Span$\{\mathbf{u}, \mathbf{v}\}$

Suppose $\mathbf{u}$, $\mathbf{v}$ are nonzero, non-parallel vectors in $\mathbb{R}^2$ (or $\mathbb{R}^3$).

Let $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$ belong to Span$\{\mathbf{u}, \mathbf{v}\}$.

We have just seen that as b varies, $b\mathbf{v}$ creates a line through the origin.

The geometric effect of adding $a\mathbf{u}$ is to translate this line in the direction of $\mathbf{u}$.

So as we vary both a and b, the vector $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$ draws out a series of parallel lines, all passing through a common line.

That is, Span$\{\mathbf{u}, \mathbf{v}\}$ describes the plane through the origin containing both $\mathbf{u}$ and $\mathbf{v}$.

Remark

Because $c\mathbf{0} = \mathbf{0}$ for every scalar c, we find that

$$\text{Span}\{\mathbf{0}\} = \{\mathbf{0}\}.$$

That is, the span of the zero vector is just the zero vector.

And because $\mathbf{0} + \mathbf{v} = \mathbf{v}$, this also shows that

$$\text{Span}\{\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\},$$

i.e. we can always remove zero vectors without changing the span.