Applications of Linear Systems

Ryan C. Daileda

Trinity University

Linear Algebra

Introduction

Today we will apply the theory and techniques we have developed for solving linear systems to a number of different problems.

We will primarily be interested in obtaining and understanding solutions in specific situations.

We will return to several of these problems later when we have more advanced tools at our disposal.

Balancing a Chemical Equation

Example 1

Limestone ($CaCO_3$) neutralizes hydronium (H_3O) in acid rain through the chemical reaction

$$H_3O + CaCO_3 \to H_2O + Ca + CO_2$$

Balance this chemical equation.

Solution. Let n_1, n_2, n_3, n_4, n_5 be the amounts of each reactant (in the order given).

Each molecule in the reaction can be represented by a vector listing the number of atoms of each element that are present.

If we order the elements by atomic weight (H, C, O, Ca), the chemical reaction yields the vector equation

$$n_1 \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + n_2 \begin{pmatrix} 0 \\ 1 \\ 3 \\ 1 \end{pmatrix} = n_3 \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + n_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + n_5 \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}.$$

This is equivalent to the homogeneous equation $A\mathbf{n} = \mathbf{0}$, where

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 3 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 3 & -1 & 0 & -2 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix}.$$

Row reduction yields

$$\begin{pmatrix} 3 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 3 & -1 & 0 & -2 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

so that n_5 is free and

$$\begin{aligned} n_1 &= 2n_5, \\ n_2 &= n_5, \\ n_3 &= 3n_5, \\ n_4 &= n_5. \end{aligned}$$

Choosing $n_5 = 1$ (the smallest possible value) we have the balanced equation

$$2H_3O + CaCO_3 \to 3H_2O + Ca + CO_2.$$

Antidifferentiation

Example 2

Compute $\int (x^3 - 2x)e^x \, dx$.

Solution. Based on experience, we assume the antiderivative has the form

$$\int (x^3 - 2x)e^x \, dx = (\underbrace{a_3x^3 + a_2x^2 + a_1x + a_0}_{p(x)})e^x + C.$$

Differentiating both sides yields

$$\begin{aligned}(x^3 - 2x)e^x &= p(x)e^x + p'(x)e^x \\ &= (a_3x^3 + (3a_3 + a_2)x^2 + (2a_2 + a_1)x + a_1 + a_0)e^x.\end{aligned}$$

Cancelling e^x and comparing coefficients of both sides gives us

$$\begin{array}{cccccc} & & & a_3 & = 1, \\ & & a_2 & +3a_3 & = 0, \\ & a_1 & +2a_2 & & = -2, \\ a_0 & +a_1 & & & = 0. \end{array}$$

Row reducing the augmented matrix we have

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 1 & 2 & 0 & -2 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

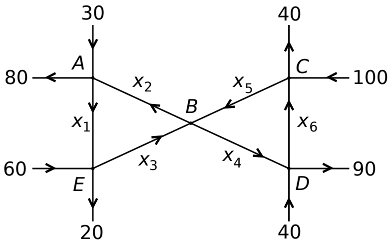
Thus $a_0 = -4$, $a_1 = 4$, $a_2 = -3$ and $a_3 = 1$, so that

$$\int (x^3 - 2x)e^x\,dx = \boxed{(x^3 - 3x^2 + 4x - 4)e^x + C}.$$

□

Network Flow

Consider the network with flow pattern shown below:



The arrows along each segment indicate flow direction in the amount indicated. The total flow into every node must equal the total flow out.

Example 3

(a) Find the general flow pattern of the network.

(b) Assuming all flows are nonnegative, what are the smallest possible values of x_2, x_3, x_4 and x_5.?

Solution. Working node by node we have:

$$
\begin{array}{ll}
\text{A:} & x_2 + 30 = x_1 + 80 \\
\text{B:} & x_3 + x_5 = x_2 + x_4 \\
\text{C:} & x_6 + 100 = x_5 + 40 \\
\text{D:} & x_4 + 40 = x_6 + 90 \\
\text{E:} & x_1 + 60 = x_3 + 20
\end{array}
\Leftrightarrow
\begin{array}{rrrrrrrr}
x_1 & -x_2 & & & & & = & -50 \\
 & x_2 & -x_3 & +x_4 & -x_5 & & = & 0 \\
 & & & & x_5 & -x_6 & = & 60 \\
 & & & x_4 & & -x_6 & = & 50 \\
x_1 & & -x_3 & & & & = & -40
\end{array}
$$

This has the augmented matrix

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -50 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 1 & 0 & -1 & 0 & 0 & 0 & -40 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & -40 \\ 0 & 1 & -1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So x_3 and x_6 are free, and

$$\begin{aligned} x_1 &= x_3 - 40, \\ x_2 &= x_3 + 10, \\ x_4 &= x_6 + 50, \\ x_5 &= x_6 + 60. \end{aligned}$$

In order for x_i to be nonnegative for all i we must have $x_3 \geq 40$ and $x_6 \geq 0$.

The smallest possible values of x_2, x_3, x_4 and x_5 occur when we have equality in both cases:

$$x_2 = 50, \;\; x_3 = 40, \;\; x_4 = 50, \;\; x_5 = 60.$$

□

Polynomial Interpolation

Consider the problem of interpolating data by a polynomial.

Suppose we are given n data points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

with $x_i \neq x_j$ for all $i \neq j$.

We seek a polynomial of degree $\leq n - 1$,

$$p(X) = c_0 + c_1 X + \cdots + c_{n-1} X^{n-1},$$

so that

$$p(x_i) = y_i \quad \text{for every} \quad i.$$

This gives us n equations

$$c_0 + c_1 x_i + c_2 x_i^2 + \cdots + c_{n-1} x_i^{n-1} = y_i, \;\; 1 \le i \le n$$

in the n variables $c_0, c_1, \ldots, c_n$.

The coefficient matrix of this system is the *Vandermonde matrix*

$$V = (x_i^{j-1}) = \begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}.$$

We will (eventually) show that a Vandermonde matrix always has a pivot in each row and column.

Thus, the system

$$A\mathbf{c} = \mathbf{y}, \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix},$$

has a unique solution for **c** which can be found by row reducing the augmented matrix $\begin{pmatrix} V & \mathbf{y} \end{pmatrix}$.

Therefore there is a *unique* polynomial of degree $\leq n-1$ that passes through our data points.

Example 4

Find the interpolating polynomial of the points $(1,-1)$, $(2,1)$, $(3,-2)$, $(4,1)$.

Solution. We set up and row reduce the Vandermonde system:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & -1 \\ 1 & 2 & 4 & 8 & 1 \\ 1 & 3 & 9 & 27 & -2 \\ 1 & 4 & 16 & 64 & 1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & 0 & -19 \\ 0 & 1 & 0 & 0 & 89/3 \\ 0 & 0 & 1 & 0 & -27/2 \\ 0 & 0 & 0 & 1 & 11/6 \end{pmatrix}.$$

So the interpolating polynomial is

$$\boxed{p(X) = -19 + \frac{89}{3}X - \frac{27}{2}X^2 + \frac{11}{6}X^3}.$$

□

Probabilities

Suppose we have an object that can be in one of n states with probabilities $P_1, P_2, \dots, P_n$, so that

$$P_1 + P_2 + \cdots + P_n = 1.$$

Furthermore, if the object is in state i, there is a *transition probability* $Q_{i \to j}$ that it will transition to state j. We assume that

$$Q_{i\to 1} + Q_{i\to 2} + \cdots + Q_{i\to n} = 1.$$

Together these probabilities satisfy

$$P_i = P_1 Q_{1\to i} + P_2 Q_{2\to i} + \cdots + P_n Q_{n\to i}, \ \ 1 \le i \le n.$$

In vector form:

$$\begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{pmatrix} = \begin{pmatrix} P_1 Q_{1\to 1} + P_2 Q_{2\to 1} + \cdots + P_n Q_{n\to 1} \\ P_1 Q_{1\to 2} + P_2 Q_{2\to 2} + \cdots + P_n Q_{n\to 2} \\ \vdots \\ P_1 Q_{1\to n} + P_2 Q_{2\to n} + \cdots + P_n Q_{n\to n} \end{pmatrix}$$

$$= (Q_{j\to i}) \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{pmatrix}.$$

This is equivalent to the homogeneous system

$$(Q_{j\to i} - \delta_{ij})\mathbf{P} = \mathbf{0},$$

where δ_{ij} is the *Kronecker delta*:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

We will (eventually) see that our hypotheses on $Q_{j \to i}$ imply that

$$(Q_{j \to i} - \delta_{ij})\mathbf{P} = \mathbf{0}$$

always has a nontrivial solution, which can be scaled to be a *probability vector* (a vector whose entries sum to 1).

Example 5

Consider the system with transition matrix

$$(Q_{j \to i}) = \begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.3 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.7 \end{pmatrix}.$$

Find the probability P_i of being in state $i = 1, 2, 3$.

Solution. We row reduce $(Q_{j\to i} - \delta_{ij})$:

$$\begin{pmatrix} -0.4 & 0.1 & 0.1 \\ 0.3 & -0.2 & 0.2 \\ 0.1 & 0.1 & -0.3 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & -0.8 \\ 0 & 1 & -2.2 \\ 0 & 0 & 0 \end{pmatrix}.$$

This yields

$$\mathbf{P} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = P_3 \begin{pmatrix} 0.8 \\ 2.2 \\ 1 \end{pmatrix}.$$

To get a probability vector we set

$$P_3(0.8 + 2.2 + 1) = 1 \;\Rightarrow\; P_3 = \frac{1}{4}.$$

Thus

$$\boxed{\mathbf{P} = \begin{pmatrix} 1/5 \\ 11/20 \\ 1/4 \end{pmatrix}}.$$

□

Electrical Networks

Ohm's Law states that, as current flows through a resistance in an electrical circuit, the difference in the electric potential between the ends of the resistance (the "voltage drop") is proportional to the current flowing through it. That is

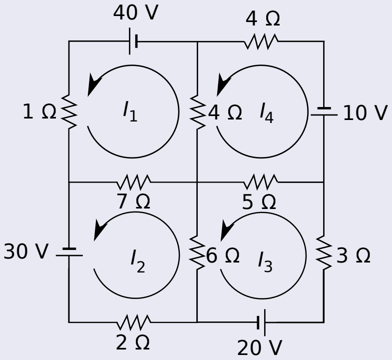
$$V = IR,$$

where V is the electrical potential (in volts), I is the current (in amperes), and R is the resistance (in ohms).

Kirchhoff's Voltage Law states that the sum of the voltage drops around a loop in a circuit is equal to the sum of the voltage sources in that loop.

Example 6

Determine the *loop currents* in the following circuit.



Solution. We apply Ohm's and Kirchhoff's Laws to each loop.

This yields the system

$$\begin{aligned} I_1 + 7(I_1 - I_2) + 4(I_1 - I_4) &= 40, \\ 2I_2 + 6(I_2 - I_3) + 7(I_2 - I_1) &= 30, \\ 3I_3 + 5(I_3 - I_4) + 6(I_3 - I_2) &= 20, \\ 4I_4 + 4(I_4 - I_1) + 5(I_4 - I_3) &= -10. \end{aligned}$$

Collecting terms with common variables this becomes

$$\begin{aligned} 12I_1 - 7I_2 \qquad\qquad - 4I_4 &= 40, \\ -7I_1 + 15I_2 - 6I_3 \qquad\quad &= 30, \\ -6I_2 + 14I_3 - 5I_4 &= 20, \\ -4I_1 \qquad\qquad - 5I_3 + 13I_4 &= -10. \end{aligned}$$

We construct the augmented matrix and row reduce:

$$\begin{pmatrix} 12 & -7 & 0 & -4 & 40 \\ -7 & 15 & -6 & 0 & 30 \\ 0 & -6 & 14 & -5 & 20 \\ -4 & 0 & -5 & 13 & -10 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & 0 & 11.43 \\ 0 & 1 & 0 & 0 & 10.55 \\ 0 & 0 & 1 & 0 & 8.04 \\ 0 & 0 & 0 & 1 & 5.84 \end{pmatrix}.$$

Thus the loop currents are

$$\boxed{I_1 = 11.43 \text{ A}, I_2 = 10.55 \text{ A}, I_3 = 8.04 \text{ A}, I_4 = 5.84 \text{ A}}.$$

□

Question. Is it surprising that this system has a unique solution?