# Linear Independence

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# Recall

We defined matrix-vector multiplication by

$$A\mathbf{x} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

This enables us to express the linear system with augmented matrix $\begin{pmatrix} A & \mathbf{b} \end{pmatrix}$ as the matrix equation

$$A\mathbf{x} = \mathbf{b}.$$

The existence of a solution to this system is equivalent to the statement that $\mathbf{b}$ is a linear combination of the columns of $A$.

That is, $\mathbf{b}$ belongs to the *column space* of $A$:

$$\mathbf{b} \in \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \text{Col}\, A.$$

The notion of *linear independence* is to uniqueness of solutions of linear systems as the notion of span is to existence.

**Definition**

Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subset \mathbb{R}^n$. We say that the vectors in $\mathcal{S}$ are *linearly independent* iff the *only* solution to

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$$

is $x_1 = x_2 = \cdots = x_k = 0$. Otherwise we say that the vectors in $\mathbf{S}$ are *linearly dependent*.

Equivalently, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent iff

$$\boxed{x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k = \mathbf{0} \;\Rightarrow\; x_1 = x_2 = \cdots = x_k = 0.}$$

Negating the definition of linear independence we find that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly *dependent* iff the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$$

has a *nonzero* solution $\mathbf{x} \in \mathbb{R}^k$.

Such a nonzero solution is called a *dependence relation*.

Taking the vectors in the definition to be the columns of a matrix $A$, we find that the columns of $A$ are linearly independent iff the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

This means that we *cannot* have any free variables (non-pivot columns) when we row reduce the coefficient matrix $A$. That is, $A$ *must* have a pivot in each column.

Note that this also tells us that, when it is consistent, the inhomogeneous equation

$$A\mathbf{x} = \mathbf{b}$$

has exactly one solution for any $\mathbf{b}$. We'll come back to this fact.

## Example 1

Determine whether or not the vectors

$$\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}5\\6\\7\\8\end{pmatrix}, \begin{pmatrix}9\\10\\11\\12\end{pmatrix} \in \mathbb{R}^4$$

are linearly independent.

*Solution.* We put the vectors into a matrix and row reduce.

$$\begin{pmatrix}1 & 5 & 9\\2 & 6 & 10\\3 & 7 & 11\\4 & 8 & 12\end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix}1 & 0 & -1\\0 & 1 & 2\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}$$

Since there is not a pivot in every column, the equation $A\mathbf{x} = \mathbf{0}$ has more than one solution. So the vectors are *linearly dependent*. □

In fact, the nontrivial solutions to $A\mathbf{x} = \mathbf{0}$ are given by

$$\begin{aligned} x_1 &= x_3, \\ x_2 &= -2x_3, \\ x_3 &\text{ is free.} \end{aligned}$$

Any nonzero choice of $x_3$ yields a dependence relation among the given vectors.

Taking $x_3 = -2$, for instance, we have

$$-2\begin{pmatrix}1\\2\\3\\4\end{pmatrix} + 4\begin{pmatrix}5\\6\\7\\8\end{pmatrix} - 2\begin{pmatrix}9\\10\\11\\12\end{pmatrix} = \mathbf{0}.$$

### Example 2

Determine whether or not the vectors

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} ,1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{R}^3$$

are linearly independent.

*Solution.* We use the vectors as the columns of a matrix and row reduce:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since we have a pivot in every column, the vectors are *linearly independent*. □

The following result and its corollary show that there is an upper limit to how large a linearly independent subset of $\mathbb{R}^n$ can be.

**Theorem 1**

*Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$. If $k > n$, then $\mathcal{S}$ is linearly dependent.*

*Proof.* If $k > n$, then the $n \times k$ coefficient matrix $A = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \end{pmatrix}$ is underdetermined.

By an earlier result, this means that the system $A\mathbf{x} = \mathbf{0}$ must have infinitely many solutions (since it is necessarily consistent).

In particular, it has nonzero solutions, which means the vectors in $\mathcal{S}$ are linearly dependent. □

Since every implication is logically equivalent to its contrapositive, we immediately obtain:

**Corollary 1**

*Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$. If $\mathcal{S}$ is linearly independent, then $k \leq n$.*

In words: the size of a linearly independent set of vectors cannot exceed the number of entries in each vector.

Although we will primarily be interested in linearly independent sets, let's take a look at the property of linear dependence first.

Notice that in the dependence relation

$$-2\mathbf{v}_1 + 4\mathbf{v}_2 + 3\mathbf{v}_3 = \mathbf{0}$$

we can solve for $\mathbf{v}_1$ in terms of $\mathbf{v}_2$ and $\mathbf{v}_3$:

$$\mathbf{v}_1 = -\frac{1}{2}(-4\mathbf{v}_2 - 3\mathbf{v}_3) = 2\mathbf{v}_2 + \frac{3}{2}\mathbf{v}_3 \in \text{Span}\{\mathbf{v}_2, \mathbf{v}_3\}.$$

That is, $\mathbf{v}_1$ is a linear combination of the remaining vectors $\mathbf{v}_2$ and $\mathbf{v}_3$.

**Remark.** It is also true that $\mathbf{v}_2$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$,and that $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, but this is not always the case for linearly dependent sets of vectors.

The most we can say in general is:

**Theorem 2 (Characterization of Linearly Dependent Sets)**

*The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ are linearly dependent if and only if (at least) one of the vectors is a linear combination of the others. In fact, if $\mathbf{v}_1 \neq \mathbf{0}$, then there is a $j > 1$ so that $\mathbf{v}_j \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}\}$.*

*Proof.* Because this is an "if and only if" statement, we must show that the two hypotheses imply each other.

Suppose that $\mathbf{v}_j$ is a linear combination of the other vectors:

$$\mathbf{v}_j = \sum_{i \neq j} c_i \mathbf{v}_i.$$

Then

$$\mathbf{v}_j - \sum_{i \neq j} c_i \mathbf{v}_i = \mathbf{0}$$

is a dependence relation among the vectors, since the coefficient of $\mathbf{v}_j$ is $1 \neq 0$. Thus the vectors are linearly dependent.

Now for the converse. Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent.

This means and that

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$$

has a nontrivial solution. Let $x_j$ be any nonzero weight.

Then

$$x_j\mathbf{v}_j = -\sum_{i\neq j} x_i\mathbf{v}_i \;\;\Rightarrow\;\; \mathbf{v}_j = \sum_{i\neq j}\frac{-x_i}{x_j}\mathbf{v}_i$$

since $x_j \neq 0$. This expresses $\mathbf{v}_j$ as a linear combination of the other vectors.

Now suppose we know that $\mathbf{v}_1 \neq 0$. Let $x_j$ be the nonzero weight with the *largest subscript*.

Then $x_{j+1} = x_{j+2} = \cdots = x_k = 0$.

If $j = 1$ this means

$$\mathbf{0} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k = x_1\mathbf{v}_1 \;\;\Rightarrow\;\; \mathbf{v}_1 = \mathbf{0},$$

since $x_1 \neq 0$. But this contradicts our hypothesis.

So $j > 1$ and

$$\mathbf{0} = \sum_{1 \le i \le k} x_i \mathbf{v}_i = \sum_{i \le j} x_i \mathbf{v}_i = \sum_{i < j} x_i \mathbf{v}_i + x_j \mathbf{v}_j.$$

As above, because $x_j \neq 0$ we can solve this for $\mathbf{v}_j$:

$$\mathbf{v}_j = \sum_{i<j} \frac{-x_i}{x_j} \mathbf{v}_i \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}\}.$$

□

Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$ be a finite (ordered) list of vectors.

Since $\mathbf{0}$ is a linear combination of any given vectors (just use zero weights), the theorem tells us that

$$\boxed{\mathbf{0} \in \mathcal{S} \;\; \Rightarrow \;\; \mathcal{S} \text{ is linearly dependent.}}$$

In particular

$$\{\mathbf{0}\} \text{ is linearly dependent.}$$

If $\mathbf{v}_j = \mathbf{v}_i$ for some $i \neq j$, then clearly $\mathbf{v}_j$ is a linear combination of the other vectors. Thus

$$\boxed{\mathbf{v}_j = \mathbf{v}_i \text{ for some } i \neq j \;\; \Rightarrow \;\; \mathcal{S} \text{ is linearly dependent.}}$$

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ and suppose that $\mathbf{v}_j$ is a linear combination of the other vectors:

$$\mathbf{v}_j = \sum_{i \neq j} c_i \mathbf{v}_i.$$

Let

$$\mathbf{x} = \sum_i d_i \mathbf{v}_i \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}.$$

Then

$$\begin{aligned}\mathbf{x} = d_j \mathbf{v}_j + \sum_{i \neq j} d_i \mathbf{v}_i &= d_j \sum_{i \neq j} c_i \mathbf{v}_i + \sum_{i \neq j} d_i \mathbf{v}_i \\ &= \sum_{i \neq j} (d_j c_i + d_i) \mathbf{v}_i \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \widehat{\mathbf{v}_j}, \ldots, \mathbf{v}_k\},\end{aligned}$$

where the "hat" indicates that $\mathbf{v}_j$ is to be omitted from the list.

This shows that every vector in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ lies in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \widehat{\mathbf{v}_j}, \ldots, \mathbf{v}_k\}$.

The reverse statement is also true, since every vector that is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \widehat{\mathbf{v}_j}, \ldots, \mathbf{v}_k$ is necessarily a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ (just take the $\mathbf{v}_j$ weight to be zero).

This proves:

**Theorem 3**

*Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. If $\mathbf{v}_j$ is a linear combination of the other vectors, then*

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \widehat{\mathbf{v}_j}, \ldots, \mathbf{v}_k\}.$$

# Example

Consider the reduced matrix

$$A = \begin{pmatrix} 1 & -2 & 0 & 1 & 4 & 0 & 0 & -3 \\ 0 & 0 & 1 & -1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{a}_7 & \mathbf{a}_8 \end{pmatrix}.$$

Notice that the nonpivot columns are linear combinations of the pivot columns to the left.

By Theorem 3, this means

$$\operatorname{Col} A = \operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6, \mathbf{a}_7, \mathbf{a}_8\} = \operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_6, \mathbf{a}_7\}.$$

We can use Theorem 3 to devise an algorithm for constructing linearly independent spanning sets.

Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$.

1. If $\mathcal{S}$ is linearly dependent, choose a vector $\mathbf{v}$ that is a linear combination of the others. Let

$$\mathcal{S}' = \mathcal{S} \setminus \{\mathbf{v}\}.$$

   Note that Theorem 3 tells us that

$$\operatorname{Span} \mathcal{S} = \operatorname{Span} \mathcal{S}'.$$

2. Repeat step 1 with $\mathcal{S}$ replaced by $\mathcal{S}'$.

Since we cannot remove vectors from $\mathcal{S}$ indefinitely, at some point we are guaranteed to have $\mathcal{S}'$ be linearly independent.

This proves the following important result.

**Theorem 4**

*Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a finite list of vectors. There is a subset $\mathcal{S}' \subseteq \mathcal{S}$ so that:*

1. *$\mathcal{S}$' is linearly independent;*
2. $\operatorname{Span} \mathcal{S}' = \operatorname{Span} \mathcal{S}$.

Notice that if we apply our proof to the set $\mathcal{S} = \{\mathbf{0}\}$, in step 1 we must remove $\mathbf{0}$, so that

$$\mathcal{S}' = \varnothing.$$

The set $\varnothing$ is considered to be linearly independent, since the "for every" condition defining linear independence is "vacuously true."

So Theorem 4 holds in this case as well, provided we define

$$\operatorname{Span} \varnothing = \{\mathbf{0}\}.$$

Now suppose we have a linearly independent set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. Let $A = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \end{pmatrix}$.
Because the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, we must have

$$\operatorname{Null} A = \{\mathbf{0}\}.$$

Choose any $\mathbf{y} \in \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \operatorname{Col} A$.
Then the equation $A\mathbf{x} = \mathbf{y}$ is consistent, by definition, and its solution set has the form

$$\mathbf{x}_0 + \operatorname{Null} A = \mathbf{x}_0 + \{\mathbf{0}\} = \{\mathbf{x}_0\}.$$

In other words, for every $\mathbf{y} \in \operatorname{Col} A$, the equation $A\mathbf{x} = \mathbf{y}$ has a *unique* solution.

**Theorem 5**

*The columns of a matrix $A$ are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \operatorname{Col} A$.*

This gives the connection between linear independence of vectors and uniqueness of solutions to linear systems.

Note that the condition $\mathbf{b} \in \operatorname{Col} A$ is just another way of saying that $A\mathbf{x} = \mathbf{b}$ is consistent.

Now let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$ be a linearly independent set of vectors.

Let $A = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_k \end{pmatrix}$.

For any $y \in \operatorname{Span} \mathcal{S} = \operatorname{Col} A$, we now know that there is a *unique* $\mathbf{x} \in \mathbb{R}^k$ so that $A\mathbf{x} = \mathbf{y}$.

We call the vector $\mathbf{x}$ the *coordinate vector of* $\mathbf{y}$ *relative to* $\mathcal{S}$ (or the vector of $\mathcal{S}$*-coordinates*), and write

$$[\mathbf{y}]_{\mathcal{S}} = \mathbf{x}.$$

Equivalently,

$$A[\mathbf{y}]_{\mathcal{S}} = \mathbf{y}.$$

Note that the computation of the $\mathcal{S}$-coordinates of $\mathbf{y}$ amounts to solving the linear system with augmented matrix $\begin{pmatrix} A & \mathbf{y} \end{pmatrix}$.

## Example 3

Show that the set of vectors

$$\mathcal{S} = \left\{ \begin{pmatrix} 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 4 \\ -4 \end{pmatrix}, \begin{pmatrix} 9 \\ -7 \\ -5 \\ -2 \end{pmatrix} \right\}$$

are linearly independent and find the $\mathcal{S}$-coordinates of

$$\mathbf{y} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ -7 \end{pmatrix}.$$

*Solution.* We can achieve both aims simultaneously by row reducing the augmented matrix:

$$\begin{pmatrix} 0 & -3 & 9 & 3 \\ 2 & 1 & -7 & 1 \\ -1 & 4 & -5 & 0 \\ 1 & -4 & -2 & -7 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Because there are pivots in all of the first three columns, the set $\mathcal{S}$ is linearly independent and

$$[\mathbf{y}]_{\mathcal{S}} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

□

# Remark

Given a linearly independent set $\mathcal{S} \subseteq \mathbb{R}^n$, the $\mathcal{S}$-coordinates of a vector $\mathbf{y} \in \operatorname{Span} \mathcal{S}$ *depend on the ordering of the vectors in* $\mathcal{S}$.

For instance, suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_2 \in R^n$ are linearly independent. Let

$$\mathbf{y} = \mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3.$$

If we take $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then

$$[\mathbf{y}]_{\mathcal{S}} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

However, if we take $\mathcal{S}' = \{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_1\}$, then

$$[\mathbf{y}]_{\mathcal{S}'} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \neq [\mathbf{y}]_{\mathcal{S}}.$$

**Moral.** Any time we talk about the $\mathcal{S}$-coordinates of a vector, we will always assume that $\mathcal{S}$ is an *ordered* list of vectors.

# Properties of Coordinates

Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a linearly independent (ordered) set of vectors with $|\mathcal{S}| = m \leq n$.

The $\mathcal{S}$-coordinate operation can be thought of as a function

$$[\cdot]_{\mathcal{S}} : \operatorname{Span} \mathcal{S} \to \mathbb{R}^m.$$

This function has some very nice properties.

**Theorem 6**

*Let everything be as above. For any* $\mathbf{x}, \mathbf{y} \in \operatorname{Span} \mathcal{S}$ *and any* $c \in \mathbb{R}$*:*

1. $[\mathbf{x} + \mathbf{y}]_{\mathcal{S}} = [\mathbf{x}]_{\mathcal{S}} + [\mathbf{y}]_{\mathcal{S}}$
2. $[c\mathbf{x}]_{\mathcal{S}} = c[\mathbf{x}]_{\mathcal{S}}$

*Furthermore,* $[\cdot]_{\mathcal{S}}$ *is* one-to-one *and* onto*.*

# Terminology

Given a function $f : X \to Y$ we say:

- $X$ is the *domain* of $f$.
- $Y$ is the *codomain* of $f$.
- $\operatorname{Im} f = \{f(x) \,|\, x \in X\}$ is the *image* or *range* of $f$.

**Definition (One-to-one)**

We say that a function $f : X$ is *one-to-one* if it never sends two distinct objects to the same place. That is:

$$x \neq y \;\; \Rightarrow \;\; f(x) \neq f(y).$$

**Remark.** Equations are usually more useful than inequalities. The (equivalent) contrapositive of this one-to-one condition is

$$f(x) = f(y) \;\Rightarrow\; x = y,$$

and this is typically how one checks that $f$ is one-to-one.

**Definition (Onto)**

We say that a function $f : X \to Y$ is onto if $\operatorname{Im} f = Y$. That is, for every $y \in Y$ the equation

$$f(x) = y$$

has a solution $x \in X$.

# Proof of Theorem 6.

Theorem 6 now follows easily from the properties of the matrix-vector product.

Suppose $\mathcal{S} \subset \mathbb{R}^n$ is linearly independent.

Let $A$ be the matrix whose columns are the members of $\mathcal{S}$.

Recall that $[\mathbf{x}]_{\mathcal{S}}$ is the unique vector in $\mathbb{R}^m$ (where $m = |\mathcal{S}|$) so that

$$A[\mathbf{x}]_{\mathcal{S}} = \mathbf{x}$$

Thus

$$A[\mathbf{x}+\mathbf{y}]_{\mathcal{S}} = \mathbf{x}+\mathbf{y} = A[\mathbf{x}]_{\mathcal{S}} + A[\mathbf{y}]_{\mathcal{S}} = A\left([\mathbf{x}]_{\mathcal{S}} + [\mathbf{y}]_{\mathcal{S}}\right).$$

Uniqueness of coordinate vectors implies that

$$[\mathbf{x}+\mathbf{y}]_{\mathcal{S}} = [\mathbf{x}]_{\mathcal{S}} + [\mathbf{y}]_{\mathcal{S}}.$$

Likewise,

$$A[c\mathbf{x}]_{\mathcal{S}} = c\mathbf{x} = c(A[\mathbf{x}]_{\mathcal{S}}) = A(c[\mathbf{x}]_{\mathcal{S}}),$$

and uniqueness of coordinates implies that

$$[c\mathbf{x}]_{\mathcal{S}} = c[\mathbf{x}]_{\mathcal{S}}.$$

The $\mathcal{S}$-coordinate map is one-to-one because coordinates are unique.

The $\mathcal{S}$-coordinate map is onto, because given any $\mathbf{y} \in \mathbb{R}^m$, if we let

$$\mathbf{x} = A\mathbf{y}$$

then we have $A\mathbf{y} = \mathbf{x} = A[\mathbf{x}]_{\mathcal{S}}$, and uniqueness of coordinates implies that

$$[\mathbf{x}]_{\mathcal{S}} = \mathbf{y}.$$

□

To demonstrate the final part of the proof, suppose we are given the linearly independent set

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

To find a vector whose $\mathcal{S}$-coordinates are $(1, 2, 3)$, we simply set

$$\mathbf{x} = 1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix}.$$

Theorem 6 will be our main tool for studying linearly independent sets.

Our first application is:

### Theorem 7

*Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a linearly independent (ordered) list of vectors with $|\mathcal{S}| = m \leq n$. Suppose $\mathcal{S}' \subseteq \operatorname{Span} \mathcal{S}$. Then $\mathcal{S}'$ is linearly independent if and only if the corresponding set of coordinate vectors*

$$[\mathcal{S}']_{\mathcal{S}} = \{[\mathbf{v}]_{\mathcal{S}} \,|\mathbf{v} \in \mathcal{S}\} \subseteq \mathbb{R}^m$$

*is linearly independent.*

*Proof.* Using the properties of $\mathcal{S}$-coordinates, we have

$$\left[\sum_{\mathbf{v}\in\mathcal{S}'} c_{\mathbf{v}}\mathbf{v}\right]_{\mathcal{S}} = \sum_{\mathbf{v}\in\mathcal{S}'} [c_{\mathbf{v}}\mathbf{v}]_{\mathcal{S}} = \sum_{\mathbf{v}\in\mathcal{S}'} c_{\mathbf{v}}[\mathbf{v}]_{\mathcal{S}}.$$

So

$$\sum_{\mathbf{v}\in\mathcal{S}'} c_{\mathbf{v}}[\mathbf{v}]_{\mathcal{S}} = \mathbf{0} \iff \left[\sum_{\mathbf{v}\in\mathcal{S}'} c_{\mathbf{v}}\mathbf{v}\right]_{\mathcal{S}} = \mathbf{0}$$

$$\iff \sum_{\mathbf{v}\in\mathcal{S}'} c_{\mathbf{v}}\mathbf{v} = \mathbf{0},$$

since the only vector with zero coordinates is the zero vector.

This means that there is a linear dependence among the vectors in $\mathcal{S}'$ if and only if there is a linear dependence among the coordinate vectors $[\mathbf{v}]_{\mathcal{S}}$, $\mathbf{v} \in \mathcal{S}'$.

This proves the theorem. $\square$