Linear Transformations Continued

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Linear Algebra

Recall

Every linear transformation $T : R^n \to \mathbb{R}^m$ is given by $T(\mathbf{x}) = A\mathbf{x}$, where A is the standard matrix

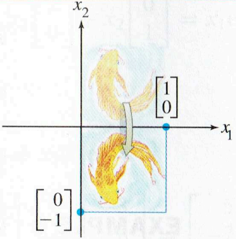
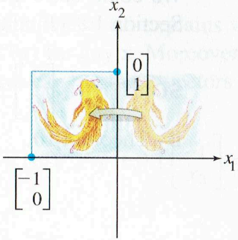
$$A = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{pmatrix}, \ \ \mathbf{e}_j = (\delta_{ij}).$$

Using this result we showed that the standard matrix for rotation in $\mathbb{R}^2$ by θ radians about the origin is

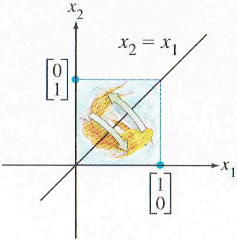
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

What about other common geometric operations on $\mathbb{R}^2$?

TABLE 1 Reflections

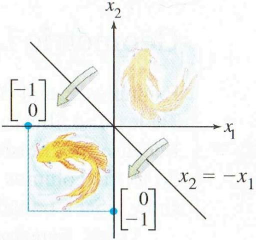
Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1-axis		$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Reflection through the x_2-axis		$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$

Reflection through the line $x_2 = x_1$



$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

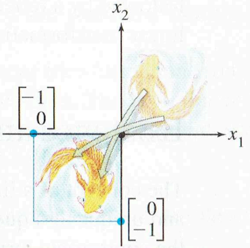
Reflection through the line $x_2 = -x_1$



$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Remark. Once we know a little matrix algebra, we will be able to compute the standard matrix for reflection across any line through the origin.

Reflection through the origin



$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

TABLE 2 Contractions and Expansions

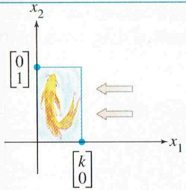
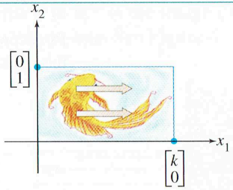
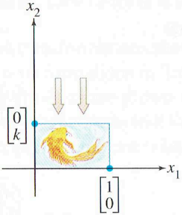
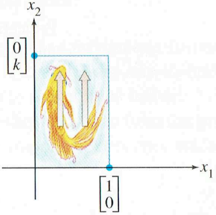
Transformation	Image of the Unit Square		Standard Matrix
Horizontal contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Vertical contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

TABLE 3 Shears

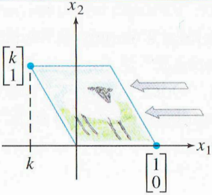
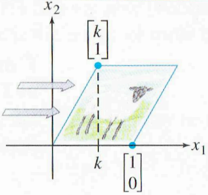
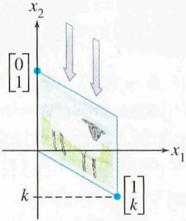
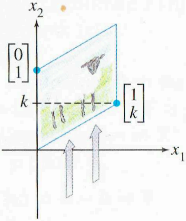
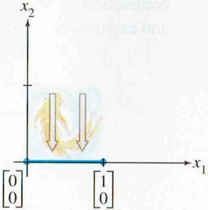
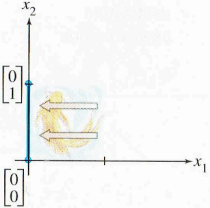
Transformation	Image of the Unit Square		Standard Matrix
Horizontal shear	 $k < 0$	 $k > 0$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear	 $k < 0$	 $k > 0$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

TABLE 4 Projections

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1-axis		$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$
Projection onto the x_2-axis		$\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

The Kernel of a Linear Transformation

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, it's *kernel* is

$$\ker T = \{\mathbf{x} \in \mathbb{R}^n \mid T(\mathbf{x}) = \mathbf{0}\}.$$

That is, $\ker T$ consists of all solutions in $\mathbb{R}^n$ to the equation $T(\mathbf{x}) = \mathbf{0}$.

If A is the standard matrix of T, then we immediately see that

$$\ker T = \operatorname{Null} A,$$

which means we can compute $\ker T$ by row reducing A.

We have seen that the solutions to $A\mathbf{x} = \mathbf{b}$ (when they exist) are always unique iff $\operatorname{Null} A = \{0\}$.

This immediately imples:

Theorem 1

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if and only if $\ker T = \{\mathbf{0}\}$.

Recall that if $A\mathbf{x}_0 = \mathbf{b}$, then *every* solution to $T(\mathbf{x}) = A\mathbf{x} = \mathbf{b}$ is given by

$$\mathbf{x}_0 + \operatorname{Null} A = \mathbf{x}_0 + \ker T.$$

Geometrically speaking, this says that the preimage of any point $\mathbf{b} \in \operatorname{im} T$ is a translation of $\ker T \leq \mathbb{R}^n$.

So ker T, in some sense, measures the failure of T to be one-to-one: the larger ker T is, the more vectors get mapped together under T.

Finally, let's state the relationship between the notions of one-to-one and onto to linear independence and spanning.

Theorem 2

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A.

1. *T is onto iff $\operatorname{Col} A = \mathbb{R}^m$ iff the columns of A span $\mathbb{R}^m$.*
2. *T is one-to-one iff $\ker T = \operatorname{Null} A = \{0\}$ iff the columns of A are linearly independent.*

Addition of Linear Transformations

If $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations, we define

$$(T + S)(\mathbf{x}) = T(\mathbf{x}) + S(\mathbf{x}).$$

Remark. This is the usual way functions would be added in calculus.

This is linear since

$$\begin{aligned}(T + S)(\mathbf{x} + \mathbf{y}) &= T(\mathbf{x} + \mathbf{y}) + S(\mathbf{x} + \mathbf{y})\\ &= T(\mathbf{x}) + T(\mathbf{y}) + S(\mathbf{x}) + S(\mathbf{y})\\ &= T(\mathbf{x}) + S(\mathbf{x}) + T(\mathbf{y}) + S(\mathbf{y})\\ &= (T + S)(\mathbf{x}) + (T + S)(\mathbf{y}),\end{aligned}$$

and similarly $(T + S)(c\mathbf{x}) = c(T + S)(\mathbf{x})$.

Scalar Multiplication of Linear Transformations

Given a scalar $c \in \mathbb{R}$ we also define

$$(cT)(\mathbf{x}) = cT(\mathbf{x}).$$

This, too, is linear:

$$\begin{aligned}(cT)(\mathbf{x}+\mathbf{y}) &= cT(\mathbf{x}+\mathbf{y}) \\ &= c(T(\mathbf{x}) + T(\mathbf{y})) \\ &= cT(\mathbf{x}) + cT(\mathbf{y}) \\ &= (cT)(\mathbf{x}) + (cT)(\mathbf{y}),\end{aligned}$$

and likewise $(cT)(a\mathbf{x}) = a(cT)(\mathbf{x})$.

The *zero transformation* is given by $\mathbf{x} \mapsto \mathbf{0} \in \mathbb{R}^m$ for all $\mathbf{x} \in \mathbb{R}^n$. We will denote it by 0.

Properties

The addition and scalar multiplication of linear transformations obeys many of the "usual" laws of arithmetic.

Theorem 3

Let S, T and U be linear transformations $\mathbb{R}^n \to \mathbb{R}^m$, and let $c, d \in \mathbb{R}$ be scalars. Then:

1. $S + T = T + S$
2. $(S + T) + U = S + (T + U)$
3. $S + 0 = 0 + S = S$
4. $c(S + T) = cS + cT$
5. $(c + d)S = cS + dS$
6. $c(dS) = (cd)S$

These all follow from the fact that vectors in $\mathbb{R}^m$ enjoy the same properties.

Suppose $S, T : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations.

Because $S + T$ is linear, it is given by a matrix.

Question. How is standard matrix of $S + T$ related to the standard matrices of S and T?

The standard matrix for $S + T$ is given by

$$\begin{aligned} &\big((S+T)(\mathbf{e}_1) \quad (S+T)(\mathbf{e}_2) \quad \cdots \quad (S+T)(\mathbf{e}_n)\big) \\ &= \big(S(\mathbf{e}_1)+T(\mathbf{e}_1) \quad S(\mathbf{e}_2)+T(\mathbf{e}_2) \quad \cdots \quad S(\mathbf{e}_n)+T(\mathbf{e}_n)\big) \end{aligned}$$

which is the matrix obtained by adding corresponding columns and the standard matrices for S and T.

Given two $m \times n$ matrices

$$A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix} = (a_{ij})$$

$$B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix} = (b_{ij})$$

we therefore define their *sum* to be

$$A + B = \begin{pmatrix} \mathbf{a}_1 + \mathbf{b}_1 & \mathbf{a}_2 + \mathbf{b}_2 & \cdots & \mathbf{a}_n + \mathbf{b}_n = (a_{ij} + b_{ij}) \end{pmatrix}.$$

Remark. This is simply the matrix obtained by adding corresponding entries in A and B.

Our work above shows that if A, B are the standard matrices for $S, T : \mathbb{R}^n \to \mathbb{R}^m$, respectively, then the standard matrix for $S + T$ is $A + B$.

Likewise, if we define

$$cA = \begin{pmatrix} c\mathbf{a}_1 & c\mathbf{a}_2 & \cdots & c\mathbf{a}_n \end{pmatrix} = (ca_{ij}),$$

and $T : \mathbb{R}^n \to \mathbb{R}^m$ has standard matrix A, then the standard matrix of cT is cA.

Theorem 3 immediately implies:

Theorem 4

Let A, B and C be $m \times n$ matrices, and let $c, d \in \mathbb{R}$ be scalars. Then:

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = 0 + A = A$
4. $c(A + B) = cA + cB$
5. $(c + d)A = cA + dA$
6. $c(dA) = (cd)A$

Here 0 denotes the $m \times n$ zero matrix, which is the standard matrix of the zero transformation.

We have

$$\begin{pmatrix} 0 & 3 & -1 \\ 4 & 0 & 7 \end{pmatrix} + \begin{pmatrix} -6 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -6 & 5 & 0 \\ 5 & 2 & 10 \end{pmatrix}$$

and

$$3 \begin{pmatrix} 1 & 2 & 3 \\ -5 & -4 & -3 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ -15 & -12 & -9 \\ 3 & 0 & -3 \end{pmatrix}.$$

Notice that for any scalar $a \in \mathbb{R}$ we have

$$\begin{pmatrix} a & & & \\ & a & & \\ & & \ddots & \\ & & & a \end{pmatrix} = a \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = aI.$$

Matrix addition and scalar multiplication interact with the matrix vector product in the way one would hope.

For instance, if A and B are $m \times n$, and $S(\mathbf{x}) = A\mathbf{x}$, $T(\mathbf{x}) = B\mathbf{x}$, then $A + B$ is the matrix of $S + T$.

Thus

$$(A + B)\mathbf{x} = (S + T)(\mathbf{x}) = S(\mathbf{x}) + T(\mathbf{x}) = A\mathbf{x} + B\mathbf{x}.$$

Scalar multiplication also interacts nicely with the matrix-vector product.

Theorem 5

Let A, B be $m \times n$ matrices, let $\mathbf{x} \in \mathbb{R}^n$ and let $a \in \mathbb{R}$. Then:

1. $(A + B)\mathbf{x} = A\mathbf{x} + B\mathbf{x}$
2. $(cA)\mathbf{x} = c(A\mathbf{x}) = A(c\mathbf{x})$

Example

Recall the rental car homework problem in which you were asked to solve the equation $A\mathbf{x} = \mathbf{x}$.

We can now write

$$A\mathbf{x} = \mathbf{x} = I\mathbf{x} \iff A\mathbf{x} - I\mathbf{x} = 0 \iff (A - I)\mathbf{x} = \mathbf{0},$$

which neatly explains the addition of -1 to the diagonal entries of A.

The more general equation $A\mathbf{x} = \lambda\mathbf{x}$ will arise in our study of eigenvalues and eigenvectors. Analogous reasoning shows that

$$A\mathbf{x} = \lambda\mathbf{x} = \lambda I\mathbf{x} \iff (A - \lambda I)\mathbf{x} = \mathbf{0},$$

which can be solved by row reducing $A - \lambda I$.

References

All diagrams were taken from

- Lay, David C., et al. *Linear Algebra and Its Applications*. 5th ed., Pearson, 2016.

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